NOTES ON THE INDUCTIVE ALGORITHM OF RESOLUTION OF SINGULARITIES BY S. ENCINAS AND O. VILLAMAYOR

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CONTENTS

CHAPTER 0. INTRODUCTION

The purpose of these notes is simply to record the regurgitation of the beautiful and elegant ideas of Encinas and Villamayor on the problem of resolution of singularties in the papers:

"A course on constructive desingularization and equivariance"

"A new theorem of desingularization over fields of characteristic zero"

"On properties of constructive desingularization" (by Encinas).

The notes are the results of seminars held at Purdue University, organized by A. Gabrielov and the author, in the Fall semester of 2000 and continued in the Spring semester of 2001.

After the first draft of these notes was written, we had the fortune of Villamayor himself visiting Purdue University to give a series of lectures titled "Constructive Desingularization". Consequently we added Chapter 6, which should explain the origin of the ingeneous t-invariant, to the revised version based upon one of his lectures. Some of the examples presented in Chapter 11 are also taken from his lectures. We thank Prof. Villamayor for his generous permission to include these in this revised version.

The following are the main themes of these notes.

Main Theme 0-1 (Resolution of singularities). *Understand the solution by Encinas and Villamayor (extending of course some of the original ideas of Hironaka) to the problem of resolution of singularities:*

Let X be a variety over a field k of characteristic zero. Establish an algorithm to construct a sequence of blowups

$$X = X_0 \overset{\pi_1}{\leftarrow} X_1 \overset{\pi_2}{\leftarrow} \cdots \overset{\pi_{l-1}}{\leftarrow} X_{l-1} \overset{\pi_l}{\leftarrow} X_l$$

so that

(i) the centers $Y_{i-1} \subset X_{i-1}$ of the blowups π_i $(i = 1, ..., l)$ are over $\mathrm{Sing}(X) = X \setminus \mathrm{Reg}(X)$,

(ii) the centers $Y_{i-1} \subset X_{i-1}$ are closed subschemes, which may be reducible and may NOT be smooth or reduced in general[1]*,*

(iii) X_l is a variety smooth over k and the induced morphism $X = X_0 \overset{\pi}{\leftarrow} X_l$, where $\pi = \pi_1 \circ \pi_2 \circ \cdots \circ \pi_{l-1} \circ \pi_l$, is a projective birational morphism isomorphic over $\mathrm{Reg}(X)$.

The main body of these notes, Chapter 1 through Chapter 7, will be devoted to the solution to the following problem of "embedded" resolution of singularities, from which the solution to the original problem of resolution of singularities immediately follows. (See Chapter 10 for detail.)

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[1] We want to emphasize that we do NOT require that the centers Y_{i-1} be smooth or even reduced, or that they be contained in the singular loci of the varieties X_{i-1}, i.e., $Y_{i-1} \subset \mathrm{Sing}(X_{i-1})$, as Hironaka or Bierstone-Milman does in their presentation of resolution of singularities. It seems that this slight weakening of the statement is a price we have to pay for dealing only with the order function and weak transforms, and not with the Hilbert-Samuel function, which is better suited for detecting the strict transforms. The author would like to thank Prof. Bierstone, who brought this fact to the attention of the author and pointed out the mistakes in the earlier manuscript.

Main Theme 0-2 (Embedded resolution of singularities). *Understand the solution to the problem of "embedded" resolution of singularities:*

Let $X \subset W$ be a variety, embedded as a closed subscheme of another variety W smooth over a field k of characteristic zero. Establish an algorithm to construct a sequence of blowups

$$X = X_0 \subset W = W_0 \overset{\pi_1}{\leftarrow} X_1 \subset W_1 \overset{\pi_2}{\leftarrow} \cdots \overset{\pi_{l-1}}{\leftarrow} X_{l-1} \subset W_{l-1} \overset{\pi_l}{\leftarrow} X_l \subset W_l$$

so that

(i) the centers $Y_{i-1} \subset W_{i-1}$ of the blowups π_i $(i = 1, ..., l)$ are over $\mathrm{Sing}(X) = X \setminus \mathrm{Reg}(X)$,

(ii) the centers $Y_{i-1} \subset W_{i-1}$[2] *are permissible with respect to the exceptional divisors $E_{i-1} \subset W_{i-1}$ for the morphisms $\psi_{i-1} = \pi_1 \circ \pi_2 \circ \cdots \circ \pi_{i-2} \circ \pi_{i-1}$ (which are simple normal crossing divisors),*

(iii) the strict transform X_l (of X_0) $\subset W_l$ is a variety smooth over k, permissible with respect to E_l, and the induced morphism $X = X_0 \overset{\pi}{\leftarrow} X_l$, where $\pi = \psi_l = \pi_1 \circ \pi_2 \circ \cdots \circ \pi_{l-1} \circ \pi_l$, is a projective birational morphism isomorphic over $\mathrm{Reg}(X)$.

Note that we say the center $Y_{i-1} \subset W_{i-1}$ is permissible with respect to E_{i-1} if Y_{i-1} are smooth, at each closed point $p \in W_{i-1}$ there exists an open neighborhood U_p with a system of regular parameters $(x_1, ..., x_d)$ such that $Y_{i-1} \cap U_p = \cap_{i \in M}\{x_i = 0\}$ and $E_{i-1} \cap U_p = \{\prod_{i \in N} x_i = 0\}$ for some subsets $M, N \subset \{1, ..., d = \dim W_{i-1}\}$, and that we say E_{i-1} is a simple normal crossing divisor where the irreducible components of E_{i-1} are required to be smooth without self-intersection, in contrast to the condition of being a normal crossing divisor where only the local requirement $E_{i-1} \cap U_p = \{\prod_{i \in N} x_i = 0\}$ is posed with the system of regular parameters $(x_1, ..., x_d)$ chosen analytically.

We remark that the solution to the problem of embedded resolution of singularities is derived from looking at our specific[3] algorithm to solve the problem of "principalization" of ideals.

Main Theme 0-3 (Principalization of ideals). *Understand the solution to the problem of "principalization" of ideals: Let W be a variety smooth over a field k of characteristic zero and $\mathcal{I} \subset \mathcal{O}_W$ be a coherent sheaf of ideals. Establish an algorithm to construct a sequence of blowups*

$$W = W_0 \overset{\pi_1}{\leftarrow} W_1 \overset{\pi_2}{\leftarrow} \cdots \overset{\pi_{l-1}}{\leftarrow} W_{l-1} \overset{\pi_l}{\leftarrow} W_l$$

[2] We want to emphasize that we do NOT require that $Y_{i-1} \subset X_{i-1}$, i.e., the centers Y_{i-1} be contained in the strict transforms X_{i-1} of $X = X_0$, or that they be contained in the singular loci of the strict transforms, i.e., $Y_{i-1} \subset \mathrm{Sing}(X_{i-1})$, as Hironaka or Bierstone-Milman does in their presentation of embedded resolution of singularities. Therefore, though the centers Y_{i-1} are smooth in the ambient varieties W_{i-1}, their restrictions $Y_{i-1} \cap X_{i-1}$ to the strict transforms may not be smooth or reduced in general.

[3] Without condition (i) imposed on our formulation of embedded resolution, which requires the centers to be taken over $\mathrm{Sing}(X)$, a solution to the problem of embedded resolution of singularities follows immediately as a corollary to the solution to the problem of principalization, if we apply the latter to the defining ideal $\mathcal{I}_X \subset \mathcal{O}_W$ and look at the stage where the strict transform becomes the center of blowup. However, in order to satisfy condition (i), we need more requirements on the algorithm of principalization. This is why we have to look at the "specific" algorithm as we discuss in these notes.

so that

(i) the centers $Y_{i-1} \subset W_{i-1}$ *of the blowups* $\pi_i \quad (i = 1, ..., l)$ *are over the support of* $\mathcal{O}_W/\mathcal{I}$*,*

(ii) the centers $Y_{i-1} \subset W_{i-1}$ *are permissible with respect to the exceptional divisors* $E_{i-1} \subset W_{i-1}$ *for the morphisms* $\psi_{i-1} = \pi_1 \circ \pi_2 \circ \cdots \circ \pi_{i-2} \circ \pi_{i-1}$*,*

(iii) the total transform $\mathcal{I}_l = \mathcal{I}\mathcal{O}_{W_l}$ *of the ideal* $\mathcal{I}$ *(We write* $\mathcal{I}\mathcal{O}_{W_l}$ *for the ideal of* $\mathcal{O}_{W_l}$ *generated by* $\psi_l^{-1}(\mathcal{I})$ *by abuse of notation.) is a product of the principal ideals defining divisors* H_j

$$\mathcal{I}_l = I(H_1)^{a_1} \cdots I(H_m)^{a_m}$$

where the divisors H_j *and the exceptional divisor* E_l *for* ψ_l *form a divisor with only simple normal crossings.*

In our formulation of the probelm of principalization, it should be emphasized and warned against the common usage of the word "principal", we require not only the (total transform of the) ideal to be locally generated by one element but also to be a product of the defining ideals of the irreducible components of a simple normal crossing divisor.

We note that, for the sake of simplicity of presentation, we assume that the base field k is algebraically closed in Chapter 1 through Chapter 7, aside from the basic assumption that k is of characteristic zero. (The general case where k may not be algebraically closed is discussed in Chapter 8 and it can be settled rather easily after the discussion of equivariance under the action of the Galois group $\mathrm{Gal}(\overline{k}/k)$ on the process prescribed over $\overline{k}$.)

The key strategy of Encinas and Villamayor is to reduce the problem of (embedded) resolution of singularities, which is reformulated as the problem of principalization, to that of resolution of singularities of "basic objects", the notion we introduce in Chapter 1. The basic objects are designed to extract the inductive nature of the problem. The elegance of their ideas is condensed in the definition of the t-invariant attached to a (sequence of) basic object(s).

Chapter 2 discusses a solution to the problem of resolution of singularities of monomial basic objects, where the given ideals (of the basic objects) are already products of the principal ideals defining the irreducible components of the boundary divisors. This turns out to be the easiest case where the solution can be given in a concise combinatorial manner. When (the maximum of) the invariant "w-ord" of a basic object is equal to 0, resolution of singularities of the basic object is reduced to that of the monomial ones. Therefore, in the later chapters, we consider an algorithm for resolution of singularities of (general) basic objects to be complete as soon as (the maximum of) the invariant w-ord is 0.

Chapter 3 reveals the key inductive lemma, which reduces the problem of resolution of singularities of a basic object of dimension d to that of resolution of singularities of charts consisting of basic objects of dimension $d-1$, and hence realizing our inductive strategy via the notion of basic objects. But there is a catch. We have to assume that the basic object to start with to be "simple" and also have to assume the existence of smooth hypersurfaces (inside of the open subsets which give rise to the charts) which cover the singularities of the simple basic object and which cross transversally with the specified boundary divisor of the original simple basic object of dimension d. The lemma forms the basis of our inductive argument.

The key inductive lemma leads us naturally to the notion of "general basic objects", generalizing the notion of basic objects, so that we can carry out the inductive argument, suggested by the lemma, in a more natural framework. This is done in Chapter 4, clarifying some minor obscure points in the original papers. As a general basic object consists of (local) charts of basic objects, there arises a problem of patching up the processes of resolution of singularities of various (local) charts of basic objects to form a unique process of resolution of singularities of the (global) general basic object. This problem will be solved by showing via Hironaka's trick that the invariants defined on individual (local) charts patch up to provide well-defined (global) invariants on the general basic object, which in turn determine the global centers of blowups in the process of resolution of singularities. This is another subject of Chapter 4.

The key inductive lemma, however, falls short of completing the inductive process of resolution of singularities of (general) basic objects for the following two reasons (difficulties) (which was described as a "catch" in the previous paragraph):

1. It requires the original basic object of dimension d to be simple, though the resulting (general) basic objects of dimension $d-1$ may not be (and in most cases actually are not) simple.

2. It requires the existence of smooth hypersurfaces (inside of the open subsets which give rise to the charts) satisfying the conditions (including the transversality) mentioned above.

The elegant and brilliant theorem of Encinas and Villamayor, discussed in Chapter 5, overcomes these two difficulties in one stroke with the use of the ingeneous t-invariant, and hence provides a complete inductive algorithm for resolution of singularities of general basic objects.

However ingeneous it may be, nonetheless, the use of the t-invariant to complete the inductive algorithm in Chapter 5 may look "slick" in the untrained eyes and seems as though it came "out of blue". Chapter 6 presents a more down-to-earth approach to the inductive algorithm, which, by decomposing the inductive algorithm into a few reduction steps, tries to explain where the t-invariant comes from and how natural it is.

The inductive algorithm for resolution of singularities of general basic objects provides a solution to the problem of principalization of ideals, achieving Main Theme 0-3. Now a solution to the problem of embedded resolution of singularities follows as an easy corollary, if we apply this specific algorithm for principalization to the defining ideal $\mathcal{I}_X \subset \mathcal{O}_W$ of an embedding $X \subset W$. The argument is presented in Chapter 7, achieving Main Theme 0-2.

We observe in Chapter 8 that the inductive algorithm is equivariant under any group action. This implies, in particular, that the process of the algorithm prescribed over $\overline{k}$, where $\overline{k}$ is the algebraic closure of the base field k, is equivariant under the action of the Galois group $\mathrm{Gal}(\overline{k}/k)$, and hence that the process is actually defined over k. This observation provides an inductive algorithm for embedded resolution of singularities over any field of characteristic zero.

In Chapter 9, we construct an invariant, based upon the w-ord, Γ- and t-invariants, of general basic objects, so that the centers of blowups in our inductive algorithm for resolution of singularities are exactly the loci where the values of this invariant attain maxima.

A variety X is covered by a finite number of open subsets U which can be embedded into smooth varieties W_U. By choosing a number d sufficiently large

and replacing W_U with $W_U \times \mathbb{A}^{d-\dim W_U}$ if necessary, we may assume that all the ambient smooth varieties W_U are of the same dimension d. We observe then that the processes of embedded resolution of singularities of $U \subset W_U$ prescribed by our inductive algorithm patch up and give rise to a sequence representing non-embedded resolution of singularities of X as stated in Main Theme 0-1. This is done in Chapter 10 via the analysis of the invariants constructed in Chapter 9.

In Chapter 11, we give examples demonstrating the mechanism and some subtleties of our inductive algorithm for embedded and non-embedded resolution of singularities.

The elementary nature of the inductive algorithm by Encinas and Villamayor, which does not even make an explicit use of the Hilbert-Samuel function and builds its key invariants upon the order (multiplicity) function, allowed us to try to make these notes self-contained. We provide complete proofs for embedded and non-embedded resolution of singularities over any field of characteristic zero (as formulated in Main Themes 0-1 and 0-2, which are slightly weaker than the formulation by Hironaka or Bierstone-Milman), with little reference to the other literature, for an easy understanding on the side of the reader. We even try to avoid referring to the original papers by Encinas and Villamayor, though almost all the proofs are taken verbatim from them.

We are very much aware of the other important developments on the subject of canonical and constructive resolution of singularities, especially the monumental paper by E. Bierstone and P. Milman:

"Canonical desingularization in characteristic zero by blowing up the maximum strata of a local invariant", Inventiones Mathematicae 128, 207-302 (1997).

The restricted attention in these notes to the inductive algorithm by Encinas and Villamayor, with no discussion on the above-mentioned developments or on the more classical papers including Hironaka's, is merely a result of the lack of resource and time to run the seminars but mainly caused by the incompetence of the author, who is responsible for any mistakes in these notes.

Note to the reader: In the process of revision, the size of these notes became much bigger than what would not scare off a reader wishing for a concise and minimal understanding of the subject. For such a reader, we would like to recommend reading only of Chapter 1 through Chapter 7 (Chapter 6 is not necessary for the logic of the development of the argument but is of great help in order to understand the core ideas behind all the technical details.), where, when he finishes, a self-contained proof for an algorithm of embedded resolution of singularities is obtained.

CHAPTER 1. BASIC OBJECTS AND INVARIANTS

In Chapter 1 through Chapter 7, the base field k is assumed to be algebraically closed and of characteristic zero.

Let W be a variety smooth over k of dimension d and $J \subset \mathcal{O}_W$ a coherent sheaf of ideals (which we simply call an ideal by abuse of language).

Definition 1-1 (Order of an ideal). *Let $p \in W$ be a point. The order $\nu_p(J)$ of an ideal $J \subset \mathcal{O}_W$ at p is defined to be*

$$\nu_p(J) := \nu(J_p) = \max\{n \in \mathbb{Z}_{\geq 0}; J_p \subset m_p^n\}$$

where m_p is the maximal ideal of the local ring $\mathcal{O}_{W,p}$ and where $J_p \subset \mathcal{O}_{W,p}$ is the stalk of J at p.

Remark 1-2 (Some properties of order).

(i) Let $\widehat{\mathcal{O}_{W,p}}$ be the (m_p-adic) completion of $\mathcal{O}_{W,p}$, $\widehat{J_p} = J_p \otimes_{\mathcal{O}_{W,p}} \widehat{\mathcal{O}_{W,p}}$ and $\widehat{m_p} = m_p \otimes_{\mathcal{O}_{W,p}} \widehat{\mathcal{O}_{W,p}}$ the completions of the ideals J_p and m_p, respectively, in $\widehat{\mathcal{O}_{W,p}} = \mathcal{O}_{W,p} \otimes_{\mathcal{O}_{W,p}} \widehat{\mathcal{O}_{W,p}}$. The order $\nu(\widehat{J_p})$ of $\widehat{J_p}$ coincides with the order $\nu_p(J) = \nu(J_p)$ of J_p, i.e.,

$$\nu(J_p) = \nu(\widehat{J_p}) = \max\{n \in \mathbb{Z}_{\geq 0}; \widehat{J_p} \subset \widehat{m_p}^n\},$$

since $\widehat{\mathcal{O}_{W,p}}$ is faithfully flat over $\mathcal{O}_{W,p}$.

Observe that $\widehat{\mathcal{O}_{W,p}}$ is isomorphic to a power series ring over k at a closed point $p \in W$, i.e., once we fix a system of regular parameters $(x_1, ..., x_d)$ we have a k-algebra isomorphism

$$\widehat{\mathcal{O}_{W,p}} \cong k[[x_1, ..., x_d]],$$

sending $x_1, ..., x_d$ of $\widehat{\mathcal{O}_{W,p}}$ to the corresponding variables in $k[[x_1, ..., x_d]]$.

Therefore,

$$\nu(\widehat{J_p}) = \min\{\nu(f); f \in \widehat{J_p}\}$$

where $\nu(f)$ is the lowest degree of the Taylor expansion of f considered as an element of the power series ring. In particular, if $\{f_i\}$ is a set of generators for J_p over $\mathcal{O}_{W,p}$ and hence for $\widehat{J_p}$ over $\widehat{\mathcal{O}_{W,p}}$, then

$$\nu(J_p) = \nu(\widehat{J_p}) = \min\{\nu(f_i)\}.$$

(ii) Let $I, J \subset \mathcal{O}_W$ be ideals. Then

$$\nu_p(I + J) = \min\{\nu_p(I), \nu_p(J)\}$$
$$\nu_p(I \cdot J) = \nu_p(I) \cdot \nu_p(J).$$

The order of an ideal can be analyzed using "derivatives". The analysis naturally leads to the following notion of the "extension" of an ideal.

Definition-Proposition 1-3 (Extension of an ideal). *Let $J \subset \mathcal{O}_W$ be an ideal.*

(i) The extension $\Delta(\widehat{J_p})$ of $\widehat{J_p} \subset \widehat{\mathcal{O}_{W,p}}$, where $p \in W$ is a closed point, is defined to be the ideal generated by the elements f of $\widehat{J_p}$ and their (partial) derivatives $\frac{\partial f}{\partial x_i}$ via k-algebra isomorphism $\widehat{\mathcal{O}_{W,p}} \cong k[[x_1, ..., x_d]]$ (cf. Remark 1-2 (i)), i.e.,

$$\Delta(\widehat{J_p}) = \langle \widehat{J_p}, \frac{\partial f}{\partial x_1}, ..., \frac{\partial f}{\partial x_d}; f \in \widehat{J_p} \rangle.$$

The extension $\Delta(\widehat{J_p})$ is determined independently of the choice of the isomorphism.

(ii) There uniquely exists an ideal $\Delta(J) \subset \mathcal{O}_W$, called the extension of J, such that

$$\Delta(J)_p \otimes_{\mathcal{O}_{W,p}} \widehat{\mathcal{O}_{W,p}} = \Delta(\widehat{J_p})$$

for all closed points $p \in W$.

Proof.

(i) It follows from the chain rule that the extension is independent of the choice of the isomorphism.

(ii) Take an affine open covering $\{U\}$ of W together with a system of regular parameters $(x_1, ..., x_d)$ over U so that $(dx_1, ..., dx_d)$ provide generators of the locally free sheaf Ω^1_W of rank d over U. Take the dual generators $(\frac{\partial}{\partial x_1}, ..., \frac{\partial}{\partial x_d})$ of the tangent sheaf T_W over U so that $(\frac{\partial}{\partial x_i}, dx_j) = \delta_{ij}$.

We only have to take the ideal generated by the elements $f \in J(U)$ and their (partial) derivatives $\frac{\partial f}{\partial x_i} = (\frac{\partial}{\partial x_i}, df)$ over U in order to define and obtain the extension $\Delta(J)|_U$.

The characterization as described in (ii) can be easily checked and implies the uniqueness at any closed point and hence of the sheaf. This also implies that the collection $\{\Delta(J)|_U\}$ patch up to provide the extension $\Delta(J)$ over W.

The relation between the order of an ideal and its extension(s) is described by the following lemma.

Lemma 1-4 (Characterization of order in terms of extensions). *Let $V(I)$ denote the zero locus of an ideal $I \subset \mathcal{O}_W$.*

(i) Let $b \in \mathbb{N}$ be a positive integer. Then

$$p \in V(\Delta^{b-1}(J)) \Longleftrightarrow \nu_p(J) \geq b$$

and hence

$$p \in V(\Delta^{b-1}(J)) \setminus V(\Delta^b(J)) \Longleftrightarrow \nu_p(J) = b,$$

where Δ^b represents the b-iterations of the operation Δ of taking the extension of an ideal.

In particular, the function $\nu_J : W \to \mathbb{Z}_{\geq 0}$ defined by $\nu_J(p) = \nu_p(J)$ is upper semi-continuous.

(ii) Let $p \in W$ be a point. Then

$$\nu_p(\Delta^{b-i}(J)) = i \Longleftrightarrow \nu_p(J) = b$$

for $i = 1, \cdots, b$.

In particular,

$$\nu_p(\Delta^{b-1}(J)) = 1 \Longleftrightarrow \nu_p(J) = b$$

Proof.

Assertions (i) and (ii) are immediate consequences of Remark 1-2 (i) for closed points. For arbitrary points, one has only to argue taking a general closed point in its closure.

We emphasize:

$\boxed{\text{The assumption of characteristic being equal to 0 is essential for this lemma.}}$

Remark 1-5 (Primitive but fundamental idea toward the inductional argument).

As a consequence of Lemma 1-4, we come to the following primitive but fundamental observation, which forms the core of our idea toward the inductive argument:

Let $J \subset \mathcal{O}_W$ be an ideal and $b_{\max} = \max\{\nu_p(J); p \in W\}$. Then the locus

$$S = \{p \in W; \nu_p(J) = b_{\max}\} = V(\Delta^{b_{\max}-1}(J))$$

is closed. For any point $p \in S$, we can find a neighborhood U_p of p in W and a smooth hypersurface $H_p \subset U_p$ such that

$$p \in S \cap U_p \subset H_p \subset U_p,$$

since $\nu_p(\Delta^{b_{\max}-1}(J)) = 1$. In fact, we have only to take an element $f_p \in \Delta^{b_{\max}-1}(J)_p$ with $\nu(f_p) = 1$, and set $H_p = \{f_p = 0\} \subset U_p$ with U_p an open neighborhood of p where f_p is regular and where the order of f_p remains 1.

This observation suggests the possibility that the analysis of the "worst" locus S of the ideal J on a d-dimensional smooth variety W may be reduced, at least locally, to the one on a $(d-1)$-dimensional smooth variety H_p, which is sometimes called a **hypersurface of maximal contact** (at p).

Now we introduce the notion of a basic object.

Definition 1-6 (Basic object). *A basic object is a triplet $(W, (J, b), E)$ where W is a variety smooth over k, (J, b) is a pair consisting of an ideal $J \subset \mathcal{O}_W$ and a positive integer $b \in \mathbb{N}$, and where $E = \{H_1, ..., H_r\}$ is a divisor with simple normal crossings. (We sometimes call E the boundary divisor of the basic object.)*

We define the singular locus of the basic object to be

$$\mathrm{Sing}(J, b) = \{p \in W; \nu_p(J) \geq b\}.$$

We call a couple (W, E), consisting of W and E as above, a pair.

Note 1-7.

We apply the same slightly abusive notation $E = \{H_1, ..., H_r\}$ in the above as was used in the original papers of Encinas and Villamayor: H_i actually consists of smooth irreducible components $H_{i,1}, ..., H_{i,l_i}$ disjoint from each other. If $i \neq j$, then H_i and H_j share no common irreducible components. We require that $\cup_{i=1,...,r,l=1,...,l_i} H_{i,l}$ is a divisor with simple normal crossings.

(Thus, strictly speaking, if we want to avoid the abuse, we should write

$$E = \{H_{1,1}, H_{1,2}, ..., H_{1,l_1}, H_{2,1}, ..., H_{2,l_2}, ..., H_{r,1}, ..., H_{r,l_r}\}.)$$

Definition 1-8 (Sequence of transformations and smooth morphisms of basic objects). *We define a sequence of transformations and smooth morphisms of basic objects*

$$\begin{aligned} (W,(J,b),E) = (W_0,(J_0,b),E_0) &\overset{\pi_1}{\leftarrow} (W_1,(J_1,b),E_1) \overset{\pi_2}{\leftarrow} \cdots \\ (W_{i-1},(J_{i-1},b),E_{i-1}) &\overset{\pi_i}{\leftarrow} (W_i,(J_i,b),E_i) \\ \cdots \overset{\pi_{k-1}}{\leftarrow} (W_{k-1},(J_{k-1},b),E_{k-1}) &\overset{\pi_k}{\leftarrow} (W_k,(J_k,b),E_k) \end{aligned}$$

to satisfy the following conditions:

Case T: $(W_{i-1},(J_{i-1},b),E_{i-1}) \overset{\pi_i}{\leftarrow} (W_i,(J_i,b),E_i)$ is a transformation.

(i) $W_{i-1} \overset{\pi_i}{\leftarrow} W_i$ is the blowup with a center $Y_{i-1} \subset W_{i-1}$ which is permissible with respect to E_{i-1}, i.e., Y_{i-1} is smooth (maybe reducible) and for any closed point $p \in W_{i-1}$ there exists an open neighborhood U_p with a system of regular parameters $(x_1, ..., x_d)$ such that

$$\begin{aligned} Y_{i-1} \cap U_p &= \cap_{m \in M} \{x_m = 0\} \\ E_{i-1} \cap U_p &= \{\prod_{m \in N} x_m = 0\} \end{aligned}$$

for some subsets $M, N \subset \{1, ..., d = \dim W_{i-1}\}$.

(ii) The center $Y_{i-1} \subset W_{i-1}$ is contained in the singular locus of the basic object $(W_{i-1},(J_{i-1},b),E_{i-1})$, i.e.,

$$Y_{i-1} \subset \mathrm{Sing}(J_{i-1}, b).$$

(iii) $H_{r+i} = \pi_i^{-1}(Y_{i-1})$ and $E_i = \{H_1, ..., H_r, H_{r+1}, ..., H_{r+i-1}, H_{r+i}\}$ where H_j $(j = 1, ..., r, r+1, ..., r+i-1)$ in E_i denotes by abuse of notation the strict transform of H_j in $E_{i-1} = \{H_1, ..., H_r, H_{r+1}, ..., H_{r+i-1}\}$.

(We also use the convention that if $H_{j,l}$ is a smooth irreducible component belonging to H_j in E_{i-1} and if $H_{j,l} \subset Y_{i-1}$, then we exclude $H_{j,l}$ from H_j in E_i and consider it as an element belonging to H_{r+i} in E_i.)

(iv) $J_i \subset \mathcal{O}_{W_i}$ is the unique ideal such that

$$J_{i-1}\mathcal{O}_{W_i} = I(H_{r+i})^b \cdot J_i.$$

(We note that the existence of such an ideal J_i, i.e., the fact that $J_{i-1}\mathcal{O}_{W_i}$ is divisible by $I(H_{r+i})^b$, is guaranteed by the condition $Y_{i-1} \subset \mathrm{Sing}(J_{i-1}, b)$ and can be checked, e.g., by Lemma 1-13.)

Case S: $(W_{i-1}, (J_{i-1}, b), E_{i-1}) \overset{\pi_i}{\leftarrow} (W_i, (J_i, b), E_i)$ *is a smooth morphism.*

(i) $W_{i-1} \overset{\pi_i}{\leftarrow} W_i$ *is a smooth morphism.*
(ii) $E_i = \{\pi_i^{-1}(H_1), ..., \pi_i^{-1}(H_r), \pi_i^{-1}(H_{r+1}), ..., \pi_i^{-1}(H_{r+i-1})\}$ *where* $E_{i-1} = \{H_1, ..., H_r, H_{r+1}, ..., H_{r+i-1}\}$. *By abuse of notation and for consistency in notaion with Case T, we write* $E_i = \{H_1, ..., H_r, H_{r+1}, ..., H_{r+i-1}, H_{r+i}\}$ *with the understanding that* $H_1, ..., H_r, H_{r+1}, ..., H_{r+i-1}$ *in* E_i *denote the corresponding pull-backs* $\pi_i^{-1}(H_1), ..., \pi_i^{-1}(H_r), \pi_i^{-1}(H_{r+1}), ..., \pi_i^{-1}(H_{r+i-1})$ *and that* $H_{r+i} = \emptyset$.
(iii) $J_i = J_{i-1}\mathcal{O}_{W_i}$.

We define a sequence of transformations and smooth morphisms of pairs

$$(W, E) = (W_0, E_0) \overset{\pi_1}{\leftarrow} (W_1, E_1) \overset{\pi_2}{\leftarrow} \cdots \overset{\pi_{k-1}}{\leftarrow} (W_{k-1}, E_{k-1}) \overset{\pi_k}{\leftarrow} (W_k, E_k)$$

to satisfy conditions (i), (iii) in Case T and (i), (ii) in Case S.

Definition 1-9 (Resolution of singularities of a basic object). *We call a sequence of transformations only (i.e., all the π_i are in Case T) of basic objects*

$$\begin{aligned} (W, (J, b), E) = (W_0, (J_0, b), E_0) &\overset{\pi_1}{\leftarrow} (W_1, (J_1, b), E_1) \overset{\pi_2}{\leftarrow} \cdots \\ (W_{i-1}, (J_{i-1}, b), E_{i-1}) &\overset{\pi_i}{\leftarrow} (W_i, (J_i, b), E_i) \\ \cdots \overset{\pi_{k-1}}{\leftarrow} (W_{k-1}, (J_{k-1}, b), E_{k-1}) &\overset{\pi_k}{\leftarrow} (W_k, (J_k, b), E_k) \end{aligned}$$

resolution of singularities of a basic object $(W, (J, b), E)$ *if*

$$\mathrm{Sing}(J_k, b) = \emptyset.$$

As will be seen in Chapter 7 through Chapter 9, the problem of (embedded) resolution of singularities, as well as the problem of principalization, can be readily reduced to the problem of resolution of singularities of (general) basic objects.

One of the keys to solve the problem of resolution of singularities of (general) basic objects is to define the following invariants ord_k, w-ord_k and t_k on a basic object $(W_k, (J_k, b), E_k)$ appearing in a sequence of transformations and smooth morphisms as above (with one extra condition on the sequence in order to define the invariant t_k).

Definition 1-10 (Key invariants of basic objects). *Let*

$$\begin{aligned} (W, (J, b), E) = (W_0, (J_0, b), E_0) &\overset{\pi_1}{\leftarrow} (W_1, (J_1, b), E_1) \overset{\pi_2}{\leftarrow} \cdots \\ (W_{i-1}, (J_{i-1}, b), E_{i-1}) &\overset{\pi_i}{\leftarrow} (W_i, (J_i, b), E_i) \\ \cdots \overset{\pi_{k-1}}{\leftarrow} (W_{k-1}, (J_{k-1}, b), E_{k-1}) &\overset{\pi_k}{\leftarrow} (W_k, (J_k, b), E_k) \end{aligned}$$

be a sequence of transformations and smooth morphisms of basic objects as defined in Definition 1-8.

(i) The invariant $\mathrm{ord}_k : \mathrm{Sing}(J_k, b) \to \frac{1}{b}\mathbb{Z}_{\geq 0}$ *is a function defined over* $\mathrm{Sing}(J_k, b)$ *such that*

$$\mathrm{ord}_k(p) = \frac{\nu_p(J_k)}{b} \textit{ for } p \in \mathrm{Sing}(J_k, b).$$

(ii) The invariant $w\text{-}\mathrm{ord}_k : \mathrm{Sing}(J_k, b) \to \frac{1}{b}\mathbb{Z}_{\geq 0}$ *is a function defined over* $\mathrm{Sing}(J_k, b)$ *such that*

$$w\text{-}\mathrm{ord}_k(p) = \frac{\nu_p(\overline{J_k})}{b} \textit{ for } p \in \mathrm{Sing}(J_k, b)$$

where $\overline{J_k} \subset \mathcal{O}_{W_k}$ *is the unique ideal characterized by*

$$J_k = I(H_{r+1})^{a_{r+1}} \cdots I(H_{r+k})^{a_{r+k}} \cdot \overline{J_k}.$$

(We note that $I(H_j)^{a_j}$ *is a multi-index notation and denotes*

$$I(H_j)^{a_j} = \prod_l I(H_{j,l})^{a_{j,l}}$$

where the $H_{j,l}$ *are the smooth irreducible components in* H_j *and where* $a_{j,l} = \nu_{\eta_{j,l}}(J_k)$ *is the order of* J_k *at the generic point* $\eta_{j,l}$ *of* $H_{j,l}$*.)*

(iii) First note that in order to define the invariant t_k *we require the following extra condition* $(\heartsuit)$ *on the sequence of transformations and smooth morphisms of basic objects:*

$$(\heartsuit) \quad \left\{ \begin{array}{l} Y_{i-1} \subset \underline{\mathrm{Max}}\ w\text{-}\mathrm{ord}_{i-1}(\subset \mathrm{Sing}(J_{i-1}, b)) \\ \textit{whenever } \pi_i \textit{ is a transformation with center } Y_{i-1} \end{array} \right\} \textit{ for } i = 1, ..., k$$

where

$$\begin{aligned} \underline{\mathrm{Max}}\ w\text{-}\mathrm{ord}_{i-1} &= \{p \in \mathrm{Sing}(J_{i-1}, b); w\text{-}\mathrm{ord}_{i-1}(p) = \max w\text{-}\mathrm{ord}_{i-1}\} \\ \max w\text{-}\mathrm{ord}_{i-1} &= \max\{w\text{-}\mathrm{ord}_{i-1}(p); p \in \mathrm{Sing}(J_{i-1}, b)\}. \end{aligned}$$

Under condition $(\heartsuit)$ *it follows that we have inequalities (See Proposition 1-12.)*

$$\begin{gathered} \max w\text{-}\mathrm{ord}_0 \geq \max w\text{-}\mathrm{ord}_1 \geq \cdots \\ \max w\text{-}\mathrm{ord}_{i-1} \geq \max w\text{-}\mathrm{ord}_i \\ \cdots \geq \max w\text{-}\mathrm{ord}_{k-1} \geq \max w\text{-}\mathrm{ord}_k. \end{gathered}$$

Let k_o *be the index so that*

$$\max w\text{-}\mathrm{ord}_{k_o-1} > \max w\text{-}\mathrm{ord}_{k_o} = \cdots = \max w\text{-}\mathrm{ord}_k.$$

(We let $k_o = 0$ *if* $\max w\text{-}\mathrm{ord}_0 = \cdots = \max w\text{-}\mathrm{ord}_k$*.) Set* $E_k = E_k^- \cup E_k^+$ *where* $E_k^- = \{H_1, ..., H_r, ..., H_{r+k_o}\}$ *as a subset of* $E_k = \{H_1, ..., H_r, ..., H_{r+k_o}, ..., H_{r+k}\}$ *and where* E_k^+ *is the complement of* E_k^- *in* E_k*. (Look at the convention explained in Definition 1-8 (iii).)*

The invariant $t_k : \mathrm{Sing}(J_k, b) \to \frac{1}{b}\mathbb{Z}_{\geq 0} \times \mathbb{Z}_{\geq 0}$ *is a function defined over* $\mathrm{Sing}(J_k, b)$ *such that*

$$t_k(p) = (w\text{-}\mathrm{ord}_k(p), n_k(p)) \textit{ for } p \in \mathrm{Sing}(J_k, b)$$

where

$$n_k(p) = \left\{ \begin{array}{l} \#\{H_i \in E_k; p \in H_i\} \textit{ if } w\text{-}\mathrm{ord}_k(p) < \max\ w\text{-}\mathrm{ord}_k \\ \#\{H_i \in E_k^-; p \in H_i\} \textit{ if } w\text{-}\mathrm{ord}_k(p) = \max\ w\text{-}\mathrm{ord}_k. \end{array} \right.$$

Remark 1-11.

(i) Both the invariants w-ord_k and t_k depend not only on the basic object $(W_k, (J_k, b), E_k)$ but also on the sequence, while the invariant ord_k is solely determined by the basic object $(W_k, (J_k, b), E_k)$.

(ii) (**Why exclude $I(H_1), ..., I(H_r)$ from the definition of $\overline{J_k}$?**) It may look more natural to define $\overline{J_k}$ to be the unique ideal so that

$$J_k = I(H_1)^{a_1} \cdots I(H_r)^{a_r} I(H_{r+1})^{a_{r+1}} \cdots I(H_{r+k})^{a_{r+k}} \cdot \overline{J_k}$$

including $I(H_1), ..., I(H_r)$ in the right hand side of the equation, especially when one realizes that this definition would make the invariant w-ord_k independent of the sequence and solely determined by the basic object $(W_k, (J_k, b), E_k)$. However, it will be clear in Chapter 4, where we introduce the notion of a general basic object, that the natural domain of definition for the invariant ord_k is the singular locus of a general basic object, which restricts to the singular loci of the basic objects forming the charts. Therefore, if we adopt the above definition allowing $I(H_1), ..., I(H_r)$ to affect $\overline{J_k}$, some of which may not lie over the singular loci, then we will not be able to conclude that w-ord_k is a well-defined invariant on the singular locus of the general basic object. More concretely, the proof in Definition-Proposition 4-5 (ii) showing that the invariant w-ord_k is determined only in terms of $\mathrm{ord}_i \quad (i = 0, ..., k)$, which are verified to be well-defined invariants on the singular loci of the general basic objects $(\mathcal{F}_i, (W_i, E_i))$ and hence that so is the invariant w-ord_k, will NOT work.

On a more historical account, $\overline{J_k}$ is called the weak transform (of the ideal $J = J_0$), in contrast to the strict transform or total transform. It seems that the letter "w" of the invariant "w-ord" comes from the word "weak" transform, indicating it is the order of the weak transform.

(iii) (**Why choose the domain to be** $\mathrm{Sing}(J_k, b)$ **and not the entire** W_k ?) As ord_k and w-ord_k are determined by the orders of the ideals J_k and $\overline{J_k}$, respectively, which are defined over the whole variety W_k, it may look artificial at this point to restrict their domains of definition to the singular locus $\mathrm{Sing}(J_k, b)$ of the basic object. It may also look unnecessary to devide the orders by the positive integer b to obtain these invariants. However, when we introduce the notion of a "general" basic object (generalizing that of a basic object) where it consists of "charts" given by many basic objects, it becomes clear that these invariants are naturally defined only over the singular locus of the general basic object, which restricts to the singular loci of the basic objects forming the charts, and that, since the positive integers b may vary from chart to chart, it is necessary to devide the orders of the ideals by these integers in order for the invariants given on the individual charts to patch. We will discuss these issues more in detail in Chapter 4.

(iv) We are NOT going to use the invariant ord_k explicitly in our algorithm for resolution of singularities, though the order (multiplicity) function is the basis of almost all of our analysis, and though ord_k and w-ord_k coincide for a sequence of transformations and smooth morphisms of simple basic objects. (See Remark 3-2 (ii) for the definition of a simple basic object.) It is used for the purpose of verifying that the invariants w-ord_k given on the individual charts, consisting of basic objects, for a general basic object patch together in Chapter 4. It is also used for the purpose of verifying that the Γ-invariant can be defined purely in terms of

the collection $\mathfrak{C}$ of sequences of transformations and smooth morphisms represented by a general basic object, free of its presentation using charts.

(v) (**Local description of the t-invariant**) The invariant t_k is indeed a local one, though superficially it depends on such global information as $\max w\text{-ord}_i$ $(i = 0, ..., k)$. In fact, under condition ($\heartsuit$) it has the following description: Let k_{op}, depending on $p \in W_k$, be the index so that

$$w\text{-ord}_{k_{op}-1}(p_{k_{op}-1}) > w\text{-ord}_{k_{op}}(p_{k_{op}}) = \cdots = w\text{-ord}_k(p_k) = w\text{-ord}_k(p)$$

where p_i is the image on W_i of $p = p_k \in W_k$. (We let $k_{op} = 0$ if $w\text{-ord}_0(p_0) = \cdots = w\text{-ord}_k(p_k) = w\text{-ord}_k(p)$.) Then

$$t_k(p) = (w\text{-ord}_k(p), n_k(p))$$

where $n_k(p)$ has the description

$$n_k(p) = \#\{H_i \in E^-_{k,p}\}$$

where $E^-_{k,p} = \{H_1, ..., H_r, H_{r+1}, ..., H_{r+k_{op}}\}$ as a subset of
$E_k = \{H_1, ..., H_r, H_{r+1}, ..., H_{r+k_{op}}, H_{r+k_{op}+1}, ..., H_{r+k}\}$, and hence $t_k(p)$ is locally determined.

In order to see that the global definition given as in Definition 1-10 (iii) and the local definition given as above coincide, one has only to observe under condition ($\heartsuit$) that

if $w\text{-ord}_k(p) < \max\ w\text{-ord}_k$, then π_i $(i = k_{op} + 1, ..., k)$ is either a smooth morphism or a transformation whose center Y_{i-1} is disjoint from p_{i-1} and hence $E_k = E^-_{k,p}$ in a neighborhood of p,

if $w\text{-ord}_k(p) = \max\ w\text{-ord}_k$, then $k_{op} \leq k_o$ and in case $k_{op} < k_o$ the morphism π_i $(i = k_{op} + 1, ..., k_o)$ is either a smooth morphism or a transformation whose center Y_{i-1} is disjoint from p_{i-1} and hence $E^-_k = E^-_{k,p}$ in a neighborhood of p.

(vi) The definition of E^-_k here is slightly different from the one in the original papers by Encinas and Villamayor, where they say "E^-_k consists of the strict transforms of the hypersurfaces (irreducible components) in E_{k_o}". After the index k_o, if we blow up along a divisor which is the strict transform of an irreducible component in E_{k_o}, the strict transform of this divisor belongs to E^-_k according to their definition, while it does not belong to E^-_k according to our definition and convention in Definition 1.8 (iii).

However, the difference between the two definitions occurs only when $w\text{-ord}_k = 0$. In our algorithm for resolution of singularities for (general) basic objects where we have condition ($\heartsuit$), as soon as (the maximum of) the invariant $w\text{-ord}_k$ is 0, we apply the method of resolution of singularities specifically prescribed for monomial (general) basic objects, where the invariant t plays no role. Thus this difference has no effect as long as our algorithm for resolution of singularities is concerned.

Our choice of the definition is only justified for the virtue of consistency with the convention in Definition 1.8 (iii) and consistency of notation for a sequence consisting both of transformations and smooth morphisms.

(vii) (**The origin of the invariant t_k**) The definition of $w\text{-ord}_k$ is natural from a view point of achieving principalization by "extracting" the exceptional divisors from the total transforms of the original ideals, though the definition of t_k may

baffle us at this point. The true ingenuity and power of the invariant t_k, especially in regard to choosing appropriate permissible centers, will be revealed in Chapter 5. A more down-to-earth explanation of the "origin" of the invariant t_k may be found in Chapter 6.

Proposition 1-12 (Properties of key invariants). *Let*

$$\begin{gathered}(W,(J,b),E) = (W_0,(J_0,b),E_0) \stackrel{\pi_1}{\leftarrow} (W_1,(J_1,b),E_1) \stackrel{\pi_2}{\leftarrow} \cdots \\ (W_{i-1},(J_{i-1},b),E_{i-1}) \stackrel{\pi_i}{\leftarrow} (W_i,(J_i,b),E_i) \\ \cdots \stackrel{\pi_{k-1}}{\leftarrow} (W_{k-1},(J_{k-1},b),E_{k-1}) \stackrel{\pi_k}{\leftarrow} (W_k,(J_k,b),E_k)\end{gathered}$$

be a sequence of transformations and smooth morphisms of basic objects.

(i) The invariants ord_k *and* $w\text{-}\mathrm{ord}_k$ *are upper semi-continuous functions.*

(ii) Note first that (by condition (ii) in Case T in Definition 1-8)

$$\mathrm{Sing}(J_{i-1},b) \supset \pi_i(\mathrm{Sing}(J_i,b)) \text{ for } i = 1,...,k.$$

Suppose that the sequence satisfies condition ($\heartsuit$)*. (See Definition 1-10 (iii).)*

Then for $i = 1,...,k$ *we have inequalities*

$$w\text{-}\mathrm{ord}_{i-1}(\xi_{i-1}) \geq w\text{-}\mathrm{ord}_i(\xi_i)$$

where $\xi_i \in \mathrm{Sing}(J_i,b)$ *and* $\xi_{i-1} = \pi_i(\xi_i) \in \mathrm{Sing}(J_{i-1},b)$*, which imply*

$$\max\ w\text{-}\mathrm{ord}_{i-1} \geq \max\ w\text{-}\mathrm{ord}_i.$$

That is to say, we have

$$\begin{gathered}\max\ w\text{-}\mathrm{ord}_0 \geq \max\ w\text{-}\mathrm{ord}_1 \geq \cdots \\ \max\ w\text{-}\mathrm{ord}_{i-1} \geq \max\ w\text{-}\mathrm{ord}_i \\ \cdots \geq \max\ w\text{-}\mathrm{ord}_{k-1} \geq \max\ w\text{-}\mathrm{ord}_k.\end{gathered}$$

The invariant t_k *is an upper semi-continuous function and for* $i = 1,...,k$ *we have inequalities*

$$t_{i-1}(\xi_{i-1}) \geq t_i(\xi_i)$$

where $\xi_i \in \mathrm{Sing}(J_i,b)$ *and* $\xi_{i-1} = \pi_i(\xi_i) \in \mathrm{Sing}(J_{i-1},b)$*, which imply*

$$\max\ t_{i-1} \geq \max\ t_i.$$

That is to say, we have

$$\begin{gathered}\max\ t_0 \geq \max\ t_1 \geq \cdots \\ \max\ t_{i-1} \geq \max\ t_i \\ \cdots \geq \max\ t_{k-1} \geq \max\ t_k.\end{gathered}$$

Proof.

(i) This is obvious, since the functions ν_{J_k} and $\nu_{\overline{J_k}}$ are upper semi-continuous functions (cf. Lemma 1-4 (i)).

(ii) First we note the following Lemma.

Lemma 1-13 (Behavior of extensions under transformation). *Let*

$$(W_0,(J_0,b),E_0) \stackrel{\pi_1}{\leftarrow} (W_1,(J_1,b),E_1)$$

be a transformation of basic objects, which is the blowup of a center $Y_0 \subset \mathrm{Sing}(J_0,b) \subset W_0$ *permissible with respect to* $E_0 = \{H_1,...,H_r\}$. *Let* $H_{r+1} = \pi_1^{-1}(Y_0)$ *be the (exceptional) divisor defined by the ideal* $I(H_{r+1}) = I(Y_0)\mathcal{O}_{W_1}$.

Then we have for $i = 0,1,...,b$

$$\Delta^{b-i}(J_0)\mathcal{O}_{W_1} \subset I(H_{r+1})^i$$
$$\frac{1}{I(H_{r+1})^i}\Delta^{b-i}(J_0)\mathcal{O}_{W_1} \subset \Delta^{b-i}(J_1).$$

Proof.

We prove the statements by decreasing induction on i.

Suppose $i = b$. Let $\eta_{r+1,l}$ be the generic point of an irreducible component $H_{r+1,l}$ in H_{r+1}, which maps onto the generic point $\theta_{0,l}$ of an irreducible component $Y_{0,l}$ of the center Y_0.

Then

$$\Delta^{b-b}(J_0)\mathcal{O}_{W_1,\eta_{r+1,l}} = J_0\mathcal{O}_{W_1,\eta_{r+1,l}} \subset m^b_{\theta_{0,l}}\mathcal{O}_{W_1,\eta_{r+1,l}} = I(H_{r+1})^b\mathcal{O}_{W_1,\eta_{r+1,l}}$$

since $Y_{0,l} \subset Y_0 \subset \mathrm{Sing}(J_0,b)$, which implies (since W_1 is nonsingular and hence factorial)

$$\Delta^{b-b}(J_0)\mathcal{O}_{W_1} = J_0\mathcal{O}_{W_1} \subset I(H_{r+1})^b.$$

Moreover, by definition we have

$$\frac{1}{I(H_{r+1})^b}\Delta^{b-b}(J_0)\mathcal{O}_{W_1} = \frac{1}{I(H_{r+1})^b}J_0\mathcal{O}_{W_1} = J_1 = \Delta^{b-b}(J_1).$$

Now suppose that the statements hold for $i \geq j$.

For any point $\xi_1 \notin H_{r+1}$, the statements for $i = j-1$ clearly hold for the stalks at the point, since π_1 is an isomorphism at ξ_1 and since $I(H_{r+1})_{\xi_1} = \mathcal{O}_{W_1,\xi_1}$.

So we have only to consider the statements for $i = j-1$ for the stalks at a closed point $\xi_1 \in H_{r+1}$.

Set $\xi_0 = \pi_1(\xi_1)$.

Consider the completions $\widehat{\mathcal{O}_{W,\xi_0}}$ and $\widehat{\mathcal{O}_{W_1,\xi_1}}$ with systems of regular parameters $(y = x_1,...,x_s,x_{s+1},...,x_d)$ and $(y,\frac{x_2}{y},...,\frac{x_s}{y},x_{s+1},...,x_d)$, where $x_1,...,x_s$ in the former ring define the center Y_0 and y in the latter ring defines the (exceptional) divisor H_{r+1}.

It suffices to show that for some generators $\{f\}$ of $\Delta^{b-(j-1)}(J_0)$ in $\widehat{\mathcal{O}_{W,\xi_0}}$, the fractions $\{\frac{f}{y^{j-1}}\}$ belong to $\Delta^{b-(j-1)}(J_1)$ in $\widehat{\mathcal{O}_{W_1,\xi_1}}$.

We take the elements f of $\Delta^{b-j}(J_0)$ and elements of the form $f = D(g)$, where $g \in \Delta^{b-j}(J_0)$ with D a k-derivation on $\widehat{\mathcal{O}_{W,\xi_0}}$, as generators of $\Delta^{b-(j-1)}(J_0)$.

Case: $f \in \Delta^{b-j}(J_0)$.

By induction we have

$$\frac{f}{y^j} \in \Delta^{b-j}(J_1) \subset \Delta^{b-(j-1)}(J_1)$$

and hence

$$\frac{f}{y^{j-1}} = y \cdot \frac{f}{y^j} \in \Delta^{b-(j-1)}(J_1).$$

Case: $f = D(g)$ where $g \in \Delta^{b-j}(J_0)$ and D is a k-derivation on $\widehat{\mathcal{O}_{W,\xi_0}} \cong k[[y = x_1, x_2, \cdots, x_s, x_{s+1}, \cdots, x_d]]$

Note first that D can be extended to a k-derivation

$$D : k[[y, \frac{x_2}{y}, \cdots, \frac{x_s}{y}, x_{s+1}, \cdots, x_d]] \to k[[y, \frac{x_2}{y}, \cdots, \frac{x_s}{y}, x_{s+1}, \cdots, x_d, \frac{1}{y}]]$$

in the obvious way (by Leibniz rule).

We claim that $D' = yD$ is a k-derivation on $\widehat{\mathcal{O}_{W_1,\xi_1}}$. In fact, it suffices to check

$$D'(y), D'(\frac{x_2}{y}), ..., D'(\frac{x_s}{y}), D'(x_{s+1}), ..., D'(x_d) \in \widehat{\mathcal{O}_{W_1,\xi_1}} \cong k[[y, \frac{x_2}{y}, \cdots, \frac{x_s}{y}, x_{s+1}, \cdots, x_d]].$$

We clearly see

$$D'(y), D'(x_{s+1}), ..., D'(x_d) \in \widehat{\mathcal{O}_{W_1,\xi_1}},$$

while for $m = 2, ..., s$ we have

$$D'(\frac{x_m}{y}) = \frac{yD(x_m) \cdot y - x_m \cdot yD(y)}{y^2} = D(x_m) - \frac{x_m}{y} D(y) \in \widehat{\mathcal{O}_{W_1,\xi_1}}.$$

Now since by induction

$$\frac{g}{y^j} \in \frac{1}{I(H_{r+1})^j} \Delta^{b-j}(J_0)\mathcal{O}_{W_1} \subset \Delta^{b-j}(J_1)(\subset \Delta^{b-(j-1)}(J_1)),$$

we have by the claim

$$D'(\frac{g}{y^j}) = \frac{D(g)}{y^{j-1}} - jD(y)\frac{g}{y^j} \in \Delta^{b-(j-1)}(J_1).$$

Therefore, we conclude finally

$$\frac{f}{y^{j-1}} = \frac{D(g)}{y^{j-1}} = D'(\frac{g}{y^j}) + jD(y)\frac{g}{y^j} \in \Delta^{b-(j-1)}(J_1),$$

completing the proof of the lemma.

We go back to the proof of Proposition 1-12 (ii).

When

$$(W_{i-1}, (J_{i-1}, b), E_{i-1}) \stackrel{\pi_i}{\leftarrow} (W_i, (J_i, b), E_i)$$

is a smooth morphism, since π is étale equivalent to the projection $W_{i-1} \leftarrow W_{i-1} \times \mathbb{A}^n$ for some n, it is obvious that

$$\frac{\nu_{\xi_{i-1}}(\overline{J_{i-1}})}{b} = \frac{\nu_{\xi}(\overline{J_i})}{b}$$

for a point $\xi_i \in W_i$ and its image $\xi_{i-1} = \pi_i(\xi_i) \in W_{i-1}$, and hence

$$\begin{aligned} \mathrm{Sing}(J_{i-1}, b) &\supset \pi_i(\mathrm{Sing}(J_i, b)) \\ w\text{-ord}_{i-1}(\xi_{i-1}) &= w\text{-ord}_i(\xi_i) \text{ for } \xi_i \in \mathrm{Sing}(J_i, b). \end{aligned}$$

Since $\max\ w\text{-ord}_{i-1} \geq \max\ w\text{-ord}_i$ and since E_i is the inverse image of E_{i-1} by the smooth morphism, we also have

$$t_{i-1}(\xi_{i-1}) = t_i(\xi_i) \text{ for } \xi_i \in \mathrm{Sing}(J_i, b).$$

Thus we have only to deal with the case where π_i is a transformation.

Since $Y_{i-1} \subset \mathrm{Sing}(J_{i-1}, b)$ as a part of the requirement for the transformation of basic objects by definition, we clearly have

$$\mathrm{Sing}(J_{i-1}, b) \supset \pi_i(\mathrm{Sing}(J_i, b)).$$

Condition $(\heartsuit)$ $Y_{i-1} \subset \underline{\mathrm{Max}}\ w\text{-ord}_{i-1}$ implies that the transformation of the basic objects

$$(W_{i-1}, (J_{i-1}, b), E_{i-1}) \stackrel{\pi_i}{\leftarrow} (W_i, (J_i, b), E_i)$$

induces another

$$(W_{i-1}, (\overline{J_{i-1}}, c), E_{i-1}) \stackrel{\pi_i}{\leftarrow} (W_i, ((\overline{J_{i-1}})_1, c), E_i),$$

where $c = b \cdot \max w\text{-ord}_{i-1}$.

Let $\xi_i \in \mathrm{Sing}(J_i, b)$ be a point and $\xi_{i-1} = \pi_i(\xi_i) \in \mathrm{Sing}(J_{i-1}, b)$. We have by Lemma 1-13

$$\begin{aligned} \Delta^{\nu_{\xi_{i-1}}(\overline{J_{i-1}})}(\overline{J_{i-1}})\mathcal{O}_{W_i} &= I(H_{r+i})^{c-\nu_{\xi_{i-1}}(\overline{J_{i-1}})} \cdot \frac{1}{I(H_{r+i})^{c-\nu_{\xi_{i-1}}(\overline{J_{i-1}})}} \Delta^{\nu_{\xi_{i-1}}(\overline{J_{i-1}})}(\overline{J_{i-1}})\mathcal{O}_{W_i} \\ &\subset I(H_{r+i})^{c-\nu_{\xi_{i-1}}(\overline{J_{i-1}})} \cdot \Delta^{\nu_{\xi_{i-1}}(\overline{J_{i-1}})}((\overline{J_{i-1}})_1) \\ &\subset \Delta^{\nu_{\xi_{i-1}}(\overline{J_{i-1}})}((\overline{J_{i-1}})_1). \end{aligned}$$

Since

$$\Delta^{\nu_{\xi_{i-1}}(\overline{J_{i-1}})}(\overline{J_{i-1}})_{\xi_{i-1}} = \mathcal{O}_{W_{i-1},\xi_{i-1}},$$

we have

$$\Delta^{\nu_{\xi_{i-1}}(\overline{J_{i-1}})}((\overline{J_{i-1}})_1)_{\xi_i} = \mathcal{O}_{W_1,\xi_i},$$

which implies

$$\nu_{\xi_{i-1}}(\overline{J_{i-1}}) \geq \nu_{\xi_i}((\overline{J_{i-1}})_1).$$

Observing $\overline{J_i} = (\overline{J_{i-1}})_1$, we conclude

$$w\text{-ord}_{i-1}(\xi_{i-1}) = \frac{\nu_{\xi_{i-1}}(\overline{J_{i-1}})}{b} \geq \frac{\nu_{\xi_i}(\overline{J_i})}{b} = w\text{-ord}_i(\xi_i).$$

We want to show that the invariant t_k is an upper semi-continuous function, i.e.,

$$F_{(\alpha,\beta)} = \{p \in \mathrm{Sing}(J_k, b); t_k(p) = (w\text{-ord}_k(p), n_k(p)) \geq (\alpha, \beta)\}$$

is closed for any $(\alpha, \beta) \in \frac{1}{b}\mathbb{Z}_{\geq 0} \times \mathbb{Z}_{\geq 0}$, where the set $\frac{1}{b}\mathbb{Z}_{\geq 0} \times \mathbb{Z}_{\geq 0}$ is given lexicographical order.

Note that the sets

$$\begin{aligned} G_\alpha &= \{p \in \mathrm{Sing}(J_k, b); w\text{-ord}_k(p) \geq \alpha\} \\ G_\alpha^+ &= \{p \in \mathrm{Sing}(J_k, b); w\text{-ord}_k(p) > \alpha\} \end{aligned}$$

are closed, since w-ord$_k$ is an upper semi-continuous function with images in $\frac{1}{b}\mathbb{Z}_{\geq 0}$.

Therefore, if $\alpha < \max\ w\text{-ord}_k$, then

$$F_{(\alpha,\beta)} = G_\alpha^+ \cup \cup_{H_{i_1},\cdots,H_{i_\beta} \in E_k} (G_\alpha \cap H_{i_1} \cap \cdots \cap H_{i_\beta}),$$

and if $\alpha = \max\ w\text{-ord}_k$, then

$$F_{(\alpha,\beta)} = G_\alpha^+ \cup \cup_{H_{i_1},\cdots,H_{i_\beta} \in E_k^-} (G_\alpha \cap H_{i_1} \cap \cdots \cap H_{i_\beta}).$$

In both cases, $F_{(\alpha,\beta)}$ is a closed subset.

Finally we show the inequality

$$t_{i-1}(\xi_{i-1}) \geq t_i(\xi_i)$$

for $\xi_i \in \mathrm{Sing}(J_i, b)$ and its image $\xi_{i-1} = \pi_i(\xi_i) \in \mathrm{Sing}(J_{i-1}, b)$.

From the first part we have

$$w\text{-ord}_{i-1}(\xi_{i-1}) \geq w\text{-ord}_i(\xi_i).$$

Suppose $w\text{-ord}_{i-1}(\xi_{i-1}) > w\text{-ord}_i(\xi_i)$. Then we obviously have

$$t_{i-1}(\xi_{i-1}) = (w\text{-ord}_{i-1}(\xi_{i-1}), n_{i-1}(\xi_{i-1}) > (w\text{-ord}_i(\xi_i), n_i(\xi_i)) = t_i(\xi_i).$$

Suppose $w\text{-ord}_{i-1}(\xi_{i-1}) = w\text{-ord}_i(\xi_i)$.

In case $\max\ w\text{-ord}_{i-1} > \max\ w\text{-ord}_i$, we have by definition $i_o = i$ and hence $E_i = E_i^-$. (See Definition 1-10 (iii) for the meaning of the number i_o.) Moreover, by condition (♡) the center $Y_{i-1} \subset \underline{\mathrm{Max}}\ w\text{-ord}_{i-1}$ is disjoint from ξ_{i-1}. Thus E_{i-1} and E_i are identical in a neighborhood of $\xi_{i-1} = \xi_i$. Therefore, we conclude

$$\begin{aligned} n_{i-1}(\xi_{i-1}) &= \#\{H \in E_{i-1}; \xi_{i-1} \in H\} \\ &= \#\{H \in E_i; \xi_i \in H\} = \#\{H \in E_i^-; \xi_i \in H\} = n_i(\xi_i) \end{aligned}$$

and hence

$$t_{i-1}(\xi_{i-1}) = t_i(\xi_i).$$

In case $\max\ w\text{-ord}_{i-1} = \max\ w\text{-ord}_i > w\text{-ord}_i(\xi_i) = w\text{-ord}_{i-1}(\xi_{i-1})$, by condition (♡) the center $Y_{i-1} \subset \underline{\mathrm{Max}}\ w\text{-ord}_{i-1}$ is disjoint from ξ_{i-1}. Thus E_{i-1} and E_i are identical in a neighborhood of $\xi_{i-1} = \xi_i$. Therefore, we conclude

$$n_{i-1}(\xi_{i-1}) = \#\{H \in E_{i-1}; \xi_{i-1} \in H\} = \#\{H \in E_i; \xi_i \in H\} = n_i(\xi_i)$$

and hence

$$t_{i-1}(\xi_{i-1}) = t_i(\xi_i).$$

In case max $w\text{-ord}_{i-1} = \max\ w\text{-ord}_i = w\text{-ord}_i(\xi_i) = w\text{-ord}_{i-1}(\xi_{i-1})$, we have by definition $(i-1)_o = i_o$. Thus the strict transforms of the divisors in E_{i-1}^- of E_{i-1} contain the divisors in E_i^- of E_i. Therefore, we conclude

$$n_{i-1}(\xi_{i-1}) = \#\{H \in E_{i-1}^-; \xi_{i-1} \in H\} \geq \#\{H \in E_i^-; \xi_i \in H\} = n_i(\xi_i)$$

and hence

$$t_{i-1}(\xi_{i-1}) \geq t_i(\xi_i).$$

This completes the proof of Proposition 1-12.

Remark 1-14.

(i) The use of Lemma 1-13, whose proof may look non-trivial, if not tricky at first, is something of an overkill just for the purpose of verifying Proposition 1-12.

For example, in order to see the inequality

$$w\text{-ord}_{i-1}(\xi_{i-1}) \geq w\text{-ord}_i(\xi_i) \text{ for } \xi_i \in \mathrm{Sing}(J_i, b) \text{ and } \xi_{i-1} = \pi_i(\xi_i) \in \mathrm{Sing}(J_{i-1}, b)$$

under the condition $Y_{i-1} \subset \underline{\mathrm{Max}}\ w\text{-ord}_{i-1}$, we have only to prove the inequality

$$\nu_q(J_0) \geq \nu_p(J_1) \text{ for } p \in W_1 \text{ and } q = \pi_1(p)$$

for a transformation of basic objects

$$(W_0, (J_0, b), E_0) \overset{\pi_1}{\leftarrow} (W_1, (J_1, b), E_1)$$

with a permissible center $Y_0 \subset \mathrm{Sing}(J_0, b)$ under the condition $b = \max \nu_{J_0}$. (See the argument in the proof above for Proposition 1-12 (ii). We may have to shrink W_{i-1} when we consider the basic object $(W_{i-1}, (\overline{J_{i-1}}, c), E_{i-1}))$ This can be seen easily using the Taylor expansion expressions of the completions of the local rings as follows:

We take the completions $\widehat{\mathcal{O}_{W_0,q}}$ and $\widehat{\mathcal{O}_{W_1,p}}$ with systems of regular parameters $(y = x_1, ..., x_s, x_{s+1}, ..., x_d)$ and $(y, \frac{x_2}{y}, ..., \frac{x_s}{y}, x_{s+1}, ..., x_d)$, where $x_1, ..., x_s$ in the former ring define the center Y_0 and y in the latter ring defines the (exceptional) divisor H_{r+1}. (Again we may assume $p \in H_{r+1}$, as the assertion is obvious otherwise.) Then

$$\begin{aligned} \widehat{\mathcal{O}_{W_0,q}} &\cong k[[y = x_1, ..., x_s, x_{s+1}, ..., x_d]], \\ \widehat{\mathcal{O}_{W_1,p}} &\cong k[[y, \frac{x_2}{y}, ..., \frac{x_s}{y}, x_{s+1}, ..., x_d]]. \end{aligned}$$

where the homomorphism $\pi_1^* : \widehat{\mathcal{O}_{W_0,q}} \to \widehat{\mathcal{O}_{W_1,p}}$ corresponds to the obvious inclusion $k[[y = x_1, ..., x_s, x_{s+1}, ..., x_d]] \hookrightarrow k[[y, \frac{x_2}{y}, ..., \frac{x_s}{y}, x_{s+1}, ..., x_d]]$. The condition $Y_0 \subset \mathrm{Sing}(J_0, b)$ translates into the statement that for any $f \in J_0$ all the monomials appearing in the Taylor expansion of f should contain $x_1, ..., x_s$ total of degree at least b. From this it follows immediately $\frac{f}{y^b} \in \widehat{\mathcal{O}_{W_1,p}}$ and J_1 is indeed well-defined.

For $f \in J_0$ with $\nu_q(f) = \nu_q(J_0) = b$, there exists in the Taylor expansion of f a monomial containing $x_1, ..., x_s$ precisely of total degree b. Then it is clear that in the Taylor expansion of $\frac{f}{y^b}$ there appears a monomial of degree $\leq b$, and hence $\nu_q(f) = b \geq \nu_p(\frac{f}{y^b}) \geq \nu_p(J_1)$.

Lemma 1-13, however, will be crucial when we define certain ideals in terms of the extensions and analyze their behavior under transformations (cf. Lemma 3-1 and Claim 3-5).

(ii) If we use the local description of the t-invariant (cf. Remark 1-11 (v)), then the proof for the statement $t_{i-1}(\xi_{i-1}) \geq t_i(\xi_i)$ becomes simpler:

As before, we have only to deal with the case where π_i is a transformation.

When $w\text{-ord}_{i-1}(\xi_{i-1}) > w\text{-ord}_i(\xi_i)$, we obviously have

$$t_{i-1}(\xi_{i-1}) = (w\text{-ord}_{i-1}(\xi_{i-1}), n_{i-1}(\xi_{i-1})) > (w\text{-ord}_i(\xi_i), n_i(\xi_i)) = t_i(\xi_i).$$

When $w\text{-ord}_{i-1}(\xi_{i-1}) = w\text{-ord}_i(\xi_i)$, we have $(i-1)_{o\xi_{i-1}} = i_{o\xi_i}$. (See Remark 1-11 (v) for the meaning of the numbers $(i-1)_{o\xi_{i-1}}$ and $i_{o\xi_i}$.) Thus the strict transforms of the divisors in $E^-_{i-1,\xi_{i-1}}$ of E_{i-1} contain the divisors in E^-_{i,ξ_i} of E_i. Therefore, we conclude

$$n_{i-1}(\xi_{i-1}) = \#\{H \in E^-_{i-1,\xi_{i-1}}; \xi_{i-1} \in H\} \geq \#\{H \in E^-_{i,\xi_i}; \xi_i \in H\} = n_i(\xi_i)$$

and hence

$$t_{i-1}(\xi_{i-1}) = (w\text{-ord}_{i-1}(\xi_{i-1}), n_{i-1}(\xi_{i-1})) \geq (w\text{-ord}_i(\xi_i), n_i(\xi_i)) = t_i(\xi_i).$$

CHAPTER 2. RESOLUTION OF SINGULARITIES OF MONOMIAL BASIC OBJECTS

In this chapter, we present an algorithm for resolution of singularities of the "monomial" basic objects. This settles the problem of resolution of singularities for any basic object (in a sequence of transformations and smooth morphisms of basic objects) with (the maximum of) the invariant w-ord (cf. Definition 1-10) being equal to 0, since it is easily reduced to the problem of resolution of singularities for some monomial basic object.

Definition 2-1 (Monomial basic object). *Let $B = (W,(J,b),E)$ be a basic object of dimension $\dim W = d$ with $E = \{H_1, ..., H_r\}$ (cf. Definition 1-6 and Note 1-7). We say B is a monomial basic object if*

$$J = I(H_1)^{a_1} \cdots I(H_r)^{a_r}.$$

(See Definition 1-10 (ii) for the meaning of the multi-index notation $I(H_j)^{a_j}$.)

Remark 2-2.

If max $w\text{-ord}_k = 0$ for a basic object $(W_k,(J_k,b),E_k)$ (in a sequence of transformations and smooth morphisms described as in Definition 1-8), then $J_k = I(H_{r+1})^{a_{r+1}} \cdots I(H_{r+k})^{a_{r+k}}$, i.e., $\overline{J_k} = \mathcal{O}_{W_k}$ in a neighborhood of the singular locus $\mathrm{Sing}(J_k,b)$. Recall that the definition of $\overline{J_k}$ only involves $I(H_{r+1}), ..., I(H_{r+k})$ but not $I(H_1), ..., I(H_r)$ (cf. Remark 1-11 (ii)).

However, once $J_k = I(H_{r+1})^{a_{r+1}} \cdots I(H_{r+k})^{a_{r+k}}$ and hence $(W_k,(J_k,b),E_k)$ is a monomial basic object (in a neighborhood of $\mathrm{Sing}(J_k,b)$), then our algorithm of resolution of singularities depends only on $(W_k,(J_k,b),E_k)$ or only on $(W_k,(J_k,b),\{H_{r+1}, ..., H_{r+k}\})$, and is independent of the sequence. This is why we characterize a monomial basic object $(W,(J,b),E)$ by the condition $J = I(H_1)^{a_1} \cdots I(H_r)^{a_r}$, involving all $I(H_1), ..., I(H_r)$.

Definition 2-3 (The Γ-invariant on a monomial basic object). *Let $(W,(J,b),E)$ be a monomial basic object of dimension $\dim W = d$ with $J = I(H_1)^{a_1} \cdots I(H_r)^{a_r}$ where $E = \{H_1, ..., H_r\}$. The invariant $\Gamma : \mathrm{Sing}(J,b) \to \mathbb{Z}_{\geq -d} \times \frac{1}{b}\mathbb{Z}_{\geq 0} \times {\mathbb{Z}_{\geq 0}}^d$ is a function defined over $\mathrm{Sing}(J,b)$ such that*

$$\Gamma(p) = (\Gamma_1(p), \Gamma_2(p), \Gamma_3(p)) \text{ for } p \in \mathrm{Sing}(J,b)$$

where

$$-\Gamma_1(p) = \min\{n; \exists j_1, ..., j_n \text{ s.t. } a_{j_1}(p) + \cdots + a_{j_n}(p) \geq b, p \in H_{j_1} \cap \cdots \cap H_{j_n}\}$$

$$\Gamma_2(p) = \max\{\frac{a_{j_1}(p) + \cdots + a_{j_n}(p)}{b}; n = -\Gamma_1(p), a_{j_1}(p) + \cdots + a_{j_n}(p) \geq b, p \in H_{j_1} \cap \cdots \cap H_{j_n}\}$$

$$\Gamma_3(p) = \max\{(j_1, ..., j_n); n = -\Gamma_1(p), \Gamma_2(p) = \frac{a_{j_1}(p) + \cdots + a_{j_n}(p)}{b}, p \in H_{j_1} \cap \cdots \cap H_{j_n}, j_1 \geq \cdots \geq j_n\}$$

with the maximum taken with respect to the lexicographical order given to ${\mathbb{Z}_{\geq 0}}^d$.

We identify $(j_1, ..., j_n)$ with $(j_1, ..., j_n, 0, ..., 0) \in {\mathbb{Z}_{\geq 0}}^d$.

We order the values of Γ *according to the lexicographical order given to* $\mathbb{Z}_{\geq -d} \times \frac{1}{b}\mathbb{Z}_{\geq 0} \times \mathbb{Z}_{\geq 0}{}^d$.

Remark 2-4.

The number $-\Gamma_1(p)$ is the minimum of the codimensions of the components given as the intersections of the hypersurfaces (in E) in $\mathrm{Sing}(J, b)$ with order (at the generic points of the components) $\geq b$ and containing the point p. We take its negaive $\Gamma_1(p) = -(-\Gamma_1(p))$ for the first factor of the invariant Γ. The moral here is:

The less the codimension is, the worse the locus is (and hence to be blown up earlier).

The number $\Gamma_2(p)$ is the maximum of the orders (devided by b) along (the generic points of) the components containing p of codimension $-\Gamma_1(p)$. The moral here is:

The more the order is, the worse the locus is (and hence to be blown up earlier).

The third factor $\Gamma_3(p)$ is the "tie breaker" given by the indices of the divisors in E. Without this third factor, two irreducible components of the maximum locus of the pair (Γ_1, Γ_2) may meet at a point p and hence not be smooth. This third factor guarantees that the maximum locus of the invariant Γ is smooth.

Proposition 2-5 (Canonical center for a monomial basic object). *Let* $B_0 = (W_0, (J_0, b), E)$ *be a monomial basic object and*

$$\begin{aligned} \underline{\mathrm{Max}}\ \Gamma_{B_0} &= \{p \in \mathrm{Sing}(J_0, b); \Gamma(p) = \max\ \Gamma_{B_0}\} \\ \max\ \Gamma_{B_0} &= \max\{\Gamma(p); p \in \mathrm{Sing}(J_0, b)\}. \end{aligned}$$

Observe that $Y_0 = \underline{\mathrm{Max}}\ \Gamma_{B_0} \subset \mathrm{Sing}(J_0, b)$ *is a smooth closed subset permissible with respect to* E_0*. Take the transformation of basic objects with center* Y_0

$$B_0 = (W_0, (J_0, b), E_0) \stackrel{\pi_1}{\leftarrow} B_1 = (W_1, (J_1, b), E_1).$$

Then B_1 *is a monomial basic object and we have*

$$\max\ \Gamma_{B_0} > \max\ \Gamma_{B_1}.$$

proof.

The proof is straightforward and left to the reader as an exercise.

Corollary 2-6 (Resolution of singularities of a monomial basic object). *Let* $(W, (J, b), E)$ *be a monomial basic object. Then there exists a sequence of transformations of monomial basic objects*

$$\begin{aligned} B_0 = (W, (J, b), E) = (W_0, (J_0, b), E_0) &\stackrel{\pi_1}{\leftarrow} B_1 = (W_1, (J_1, b), E_1) \stackrel{\pi_2}{\leftarrow} \cdots \\ B_{i-1} = (W_{i-1}, (J_{i-1}, b), E_{i-1}) &\stackrel{\pi_i}{\leftarrow} B_i = (W_i, (J_i, b), E_i) \\ \cdots \stackrel{\pi_{k-1}}{\leftarrow} B_{k-1} = (W_{k-1}, (J_{k-1}, b), E_{k-1}) &\stackrel{\pi_k}{\leftarrow} B_k = (W_k, (J_k, b), E_k) \end{aligned}$$

which represents resolution of singularities, i.e.,

$$\mathrm{Sing}(J_k, b) = \emptyset,$$

where

$$B_{i-1} = (W_{i-1}, (J_{i-1}, b), E_{i-1}) \overset{\pi_i}{\leftarrow} B_i = (W_i, (J_i, b), E_i) \quad i = 1, ..., k$$

are the transformations with centers $Y_{i-1} = \underline{\mathrm{Max}}\ \Gamma_{B_{i-1}}$.

Proof.

It follows immediately from Proposition 2-5 and the observation that the set $\mathbb{Z}_{\geq -d} \times \frac{1}{b}\mathbb{Z}_{\geq 0} \times {\mathbb{Z}_{\geq 0}}^d$ satisfies the descending chain condition (i.e., it admits no infinite strictly decreasing sequence).

Corollary 2-7 (Resolution of singularities of a basic object with $\max w\text{-ord} = 0$**).** *Let*

$$B_0 = (W, (J, b), E) = (W_0, (J_0, b), E_0) \overset{\pi_1}{\leftarrow} B_1 = (W_1, (J_1, b), E_1) \overset{\pi_2}{\leftarrow} \cdots$$
$$B_{i-1} = (W_{i-1}, (J_{i-1}, b), E_{i-1}) \overset{\pi_i}{\leftarrow} B_i = (W_i, (J_i, b), E_i)$$
$$\cdots \overset{\pi_{k-1}}{\leftarrow} B_{k-1} = (W_{k-1}, (J_{k-1}, b), E_{k-1}) \overset{\pi_k}{\leftarrow} B_k = (W_k, (J_k, b), E_k)$$

be a sequence of transformations and smooth morphisms of basic objects.

Suppose $\max w\text{-ord}_k = 0$.

Then there exists an open neighborhood $\mathrm{Sing}(J_k, b) \subset V_k \subset W_k$ *of* $\mathrm{Sing}(J_k, b)$ *such that* $(V_k, (J_k|_{V_k}, b), E_k|_{V_k})$ *is a monomial basic object.*

Take the sequence of transformations of monomial basic objects as described in Corollary 2-6

$$B_k|_{V_k} \overset{\pi_{k+1}|_{V_{k+1}}}{\longleftarrow} B_{k+1}|_{V_{k+1}} \cdots \overset{\pi_{k+N-1}|_{V_{k+N-1}}}{\longleftarrow} B_{k+N-1}|_{V_{k+N-1}} \overset{\pi_{k+N}|_{V_{k+N}}}{\longleftarrow} B_{k+N}|_{V_{k+N}}$$

with centers $Y_{i-1} = \underline{\mathrm{Max}}\ \Gamma_{B_{i-1}|_{V_{i-1}}}$ *for* $i = k+1, ..., k+N$, *which represents resolution of singularities of the monomial basic object* $B_k|_{V_k}$.

The sequence can naturally be expanded to a sequence of transformations of the original basic object $B_k = (W_k, (J_k, b), E_k)$

$$B_k = (W_k, (J_k, b), E_k) \overset{\pi_{k+1}}{\leftarrow} \cdots \overset{\pi_{k+N}}{\leftarrow} B_{k+N} = (W_{k+N}, (J_{k+N}, b), E_{k+N})$$

with the same centers Y_{i-1}, *which repesents resolution of singularities of the basic object* B_k, *i.e.,*

$$\mathrm{Sing}(J_{k+N}, b) = \emptyset.$$

Moreover, the expanded sequence is independent of the choice of the open neighborhood V_k.

Proof.

Since $\max\ w\text{-ord}_k = 0$ and since the order function $\nu_{\overline{J_k}}$ is upper semi-continuous, writing (cf. Definition 1-10 (ii))

$$J_k = I(H_{r+1})^{a_{r+1}} \cdots I(H_{r+k})^{a_{r+k}} \cdot \overline{J_k}$$

we conclude that $S = \mathrm{Supp}(\mathcal{O}_{W_k}/\overline{J_k})$ is a closed subset disjoint from $\mathrm{Sing}(J_k, b)$. Take V_k to be any open subset such that $\mathrm{Sing}(J_k, b) \subset V_k \subset W_k \setminus S$. Then $B_k|_{V_k}$ is a monomial basic object.

Remark that the centers $\underline{\mathrm{Max}}\,\Gamma_{B_i|_{V_i}}$ for the sequence of transformations decribed as in Corollary 2-6 are all over $\mathrm{Sing}(J_k, b)$ and hence that the sequence can be expanded as claimed.

As the centers chosen according to Proposition 2-5 are easily seen to be independent of the choice of V_k, so is the sequence.

We note that under the specified transformations max w-ord remains zero, i.e.,

$$\max\ w\text{-ord}_{k+i} = 0 \text{ for } i = 0, ..., k-1$$

and hence that the invariant Γ_{k+i} is well-defined.

CHAPTER 3. KEY INDUCTIVE LEMMA

In this chapter, we prove the key inductive lemma , which reduces the problem of resolution of singularities of a "simple" basic object (See Remark 3-2.) of dimension d to that of "charts" consisting of basic objects of dimension $d-1$, provided the existence of smooth hypersurfaces (in the open subsets which give rise to the charts) which cover the singularities of the simple basic object and which cross transversally with the specified boundary divisors of the original simple basic object of dimension d. This lemma will become the prototype of the inductive argument which follows, leading to the notion of general basic objects. The shortcomings of the key inductive lemma, namely the requirements for the original basic object to be simple (while the resulting basic objects of dimension $(d-1)$ in charts may not be simple) and for the existence of certain smooth hypersurfaces with the transversality condition, will be overcome in the ultimate inductive algorithm toward the general solution of resolution of singularities in Chapter 5 via the brilliant use of the t-invariant.

The underlying idea of the key inductive lemma may be most transparent when we consider an ideal $\langle f\rangle \subset k[x_1,...,x_{d-1},x_d]$ generated by an element f of the form

$$f = x_d^n + c_{n-2}x_d^{n-2} + \cdots + c_1x_d + c_0$$

where the coefficient c_{n-1} of the term x_d^{n-1} is zero after the Tshirnhausen transformation and where the coefficients $c_{n-2},...,c_0$ depend only on the variables $x_1,...,x_{d-1}$. We reduce the problem of resolution of singularities of f on $\mathbb{A}^d$ of dimension d (around the origin) to that of the coefficients $c_{n-2},...,c_1,c_0$ on $\{x_d=0\}$ of dimension $d-1$, which is a hypersurafce of maximal contact. (See Remark 1-5 for the notion of a hypersurface of maximal contact. Check that the condition $c_{d-1}=0$ immediately implies that $\{x_d=0\}$ indeed is. See Remark 3-7 for more details.)

Lemma 3-1 (Key inductive lemma). *Let $B=(W,(J,b),E)$ be a basic object with an open covering $\{W^\lambda\}_{\lambda\in\Lambda}$ satisfying the following conditions: for each $\lambda\in\Lambda$, there exists a smooth hypersurface $W_h^\lambda\subset W^\lambda$, embedded as a closed subscheme, such that*

1. $I(W_h^\lambda)\subset\Delta^{b-1}(J)|_{W^\lambda}$ (and hence $W_h^\lambda\supset \mathrm{Sing}(J,b)\cap W^\lambda$), and

2. W_h^λ is permissible with respect to $E\cap W^\lambda$, and W_h^λ is not contained in E, i.e., $W_h^\lambda\not\subset E$.

Then, $R(1)(\mathrm{Sing}(J,b))$ denoting the union of irreducible components in $\mathrm{Sing}(J,b)$ of codimension one (i.e., of dimension $\dim W-1$), we have the following:

Case A: *$R(1)(\mathrm{Sing}(J,b))\neq\emptyset$.*

In this case, the set $R(1)(\mathrm{Sing}(J,b))$ is smooth, open and closed in $\mathrm{Sing}(J,b)$ (i.e., a union of smooth connected components of $\mathrm{Sing}(J,b)$ disjoint from each other). Condition 2 guarantees that it is also permissible with respect to E.

Take

$$(W,(J,b),E)=(W_0,(J_0,b),E_0)\leftarrow(W_1,(J_1,b),E_1)$$

to be the transformation with center $Y_0=R(1)(\mathrm{Sing}(J,b))$.

Then

$$R(1)(\mathrm{Sing}(J_1,b))=\emptyset.$$

Case B: $R(1)(\mathrm{Sing}(J,b)) = \emptyset$.

In this case, let $\mathfrak{C}$ be the collection of all the sequences of transformations and smooth morphisms of pairs with specified closed subsets

$$(F_0, (W_0, E_0)) \leftarrow \cdots \leftarrow (F_k, (W_k, E_k))$$

induced from the sequences of transformations and smooth morphisms of basic objects

$$(W, (J, b), E) = (W_0, (J_0, b), E_0) \leftarrow \cdots \leftarrow (W_k, (J_k, b), E_k)$$

where the specified closed subsets are the singular loci of the corresponding basic objects, i.e.,

$$F_i = \mathrm{Sing}(J_i, b) \quad \textit{for } i = 0, 1, ..., k.$$

Then with respect to the open covering $\{W^\lambda\}_{\lambda \in \Lambda}$ we can construct the following data $\mathcal{D}_\lambda$ for each λ:

(i) $j_0^\lambda : (\widetilde{W_0^\lambda}, \widetilde{E_0^\lambda}) \hookrightarrow (W_0^\lambda, E_0^\lambda) = (W^\lambda, E_0 \cap W^\lambda)$ is an immersion of pairs, that is to say, $\widetilde{W_0^\lambda} = W_h^\lambda \hookrightarrow W^\lambda$ is a closed immersion of a $(\dim W - 1)$-dimensional smooth variety $\widetilde{W_0^\lambda}$ into W^λ, $\widetilde{W_0^\lambda}$ is permissible with respect to E_0^λ, $\widetilde{W_0^\lambda}$ is not contained in E_0^λ, i.e., $\widetilde{W_0^\lambda} \not\subset E_0^\lambda$, and $\widetilde{E_0^\lambda} = E_0^\lambda \cap \widetilde{W_0^\lambda}$,

(ii) a basic object $(\widetilde{W_0^\lambda}, (\mathfrak{a}_0^\lambda, b^\lambda), \widetilde{E_0^\lambda}) = (W_h^\lambda, (C(J^\lambda), b!), E_h^\lambda)$ where $E_h^\lambda = E \cap W_h^\lambda$ (See the proof below for the definition of the ideal $C(J^\lambda)$.),

satisfying the following conditions (GB-0,1,2,3):

(GB-0) The trivial sequence consisting only of $(F_0, (W_0, E_0))$ is in the collection $\mathfrak{C}$, i.e.,

$$(F_0, (W_0, E_0)) \in \mathfrak{C}$$

and

$$F_0 = \mathrm{Sing}(J_0, b) = \cup \mathrm{Sing}(\mathfrak{a}_0^\lambda, b^\lambda) \textit{ with } F_0 \cap W_0^\lambda = \mathrm{Sing}(\mathfrak{a}_0^\lambda, b^\lambda).$$

(GB-1) For any sequence of transformations and smooth morphisms in the collection $\mathfrak{C}$

$$(F_0, (W_0, E_0)) \leftarrow \cdots \leftarrow (F_k, (W_k, E_k))$$

there corresponds for each λ a sequence of transformations (with the same centers) and (the same) smooth morphisms (obtained by taking the Cartesian products)

$$(\widetilde{W_0^\lambda}, (\mathfrak{a}_0^\lambda, b^\lambda), \widetilde{E_0^\lambda}) \leftarrow \cdots \leftarrow (\widetilde{W_k^\lambda}, (\mathfrak{a}_k^\lambda, b^\lambda), \widetilde{E_k^\lambda})$$

with the natural immersions

$$\begin{array}{ccccc} (W_0^\lambda, E_0^\lambda) & \longleftarrow & \cdots & \longleftarrow & (W_k^\lambda, E_k^\lambda) \\ \uparrow & & & & \uparrow \\ (\widetilde{W_0^\lambda}, \widetilde{E_0^\lambda}) & \longleftarrow & \cdots & \longleftarrow & (\widetilde{W_k^\lambda}, \widetilde{E_k^\lambda}) \end{array}$$

and we have

$$F_i = \mathrm{Sing}(J_i, b) = \cup \mathrm{Sing}(\mathfrak{a}_i^\lambda, b^\lambda) \text{ with } F_i \cap W_i^\lambda = \mathrm{Sing}(\mathfrak{a}_i^\lambda, b^\lambda)$$

for $i = 0, 1, ..., k$.

(We note here that in the above clause "there corresponds ...", it is required that whenever $(W_{i-1}, E_{i-1}) \stackrel{\pi_i}{\leftarrow} (W_i, E_i)$ *is a transformation with center* $Y_{i-1} \subset W_{i-1}$*, the center* Y_{i-1} *is permissible for each* $(\widetilde{W_{i-1}^\lambda}, (\mathfrak{a}_{i-1}^\lambda, b^\lambda), \widetilde{E_{i-1}^\lambda})$*, i.e.,* $Y_{i-1} \cap W_{i-1}^\lambda \subset \widetilde{W_{i-1}^\lambda}$*,* $Y_{i-1} \cap W_{i-1}^\lambda$ *is permissible with respect to* $\widetilde{E_{i-1}^\lambda}$*, and* $Y_{i-1} \cap W_{i-1}^\lambda \subset \mathrm{Sing}(\mathfrak{a}_{i-1}^\lambda, b^\lambda)$*.)*

(GB-2) Let

$$(F_0, (W_0, E_0)) \leftarrow \cdots \leftarrow (F_k, (W_k, E_k))$$

be a sequence of transformations and smooth morphisms in $\mathfrak{C}$ *and*

$$\{(\widetilde{W_0^\lambda}, (\mathfrak{a}_0^\lambda, b^\lambda), \widetilde{E_0^\lambda}) \leftarrow \cdots \leftarrow (\widetilde{W_k^\lambda}, (\mathfrak{a}_k^\lambda, b^\lambda), \widetilde{E_k^\lambda})\}$$

the corresponding sequences (indexed by $\lambda \in \Lambda$*) of transformations and smooth morphisms as in (GB-1).*

We take a morphism of pairs $(W_k, E_k) \stackrel{\pi_{k+1}}{\leftarrow} (W_{k+1}, E_{k+1})$ *which is either in Case T or Case S.*

Case T: $(W_k, E_k) \stackrel{\pi_{k+1}}{\leftarrow} (W_{k+1}, E_{k+1})$ *is a transformation with center* $Y_k \subset W_k$*, satisfying the condition that* Y_k *is permissible for each* $(\widetilde{W_k^\lambda}, (\mathfrak{a}_k^\lambda, b^\lambda), \widetilde{E_k^\lambda})$*, i.e.,* $Y_k \cap W_k^\lambda \subset \widetilde{W_k^\lambda}$*,* $Y_k \cap W_k^\lambda$ *is permissible with respect to* $\widetilde{E_k^\lambda}$*, and* $Y_k \cap W_k^\lambda \subset \mathrm{Sing}(\mathfrak{a}_k^\lambda, b^\lambda)$.

Case S: $(W_k, E_k) \stackrel{\pi_{k+1}}{\leftarrow} (W_{k+1}, E_{k+1})$ *is a smooth morphism.*

Then we have the following assertions on the extension of the original sequence:

Case T: Take for each λ *the corresponding transformation of basic objects with center* $Y_k \cap W_k^\lambda$

$$(\widetilde{W_k^\lambda}, (\mathfrak{a}_k^\lambda, b^\lambda), \widetilde{E_k^\lambda}) \stackrel{\pi_{k+1}^\lambda}{\leftarrow} (\widetilde{W_{k+1}^\lambda}, (\mathfrak{a}_{k+1}^\lambda, b^\lambda), \widetilde{E_{k+1}^\lambda}).$$

In this case, Y_k *is permissible with respect to* $(W_k, (J_k, b), E_k)$*, i.e.,* Y_k *is permissible with respect to* E_k *and* $Y_k \subset \mathrm{Sing}(J_k, b)$*, and we have the induced transformation of basic objects*

$$(W_k, (J_k, b), E_k) \stackrel{\pi_{k+1}}{\leftarrow} (W_{k+1}, (J_{k+1}, b), E_{k+1}).$$

Then

$$F_{k+1} := \cup \mathrm{Sing}(\mathfrak{a}_{k+1}^\lambda, b^\lambda)$$

is a closed subset of W_{k+1} *with*

$$F_{k+1} \cap W_{k+1}^\lambda = \mathrm{Sing}(\mathfrak{a}_{k+1}^\lambda, b^\lambda),$$

and the extended sequence belongs to $\mathfrak{C}$*, i.e.,* $F_{k+1} = \mathrm{Sing}(J_{k+1}, b)$ *and*

$$(F_0, (W_0, E_0)) \leftarrow \cdots \leftarrow (F_k, (W_k, E_k)) \leftarrow (F_{k+1}, (W_{k+1}, E_{k+1})) \in \mathfrak{C}.$$

Case S: Take for each λ *the corresponding smooth morphism of basic objects*

$$(\widetilde{W_k^\lambda}, (\mathfrak{a}_k^\lambda, b^\lambda), \widetilde{E_k^\lambda}) \overset{\pi_{k+1}^\lambda}{\longleftarrow} (\widetilde{W_{k+1}^\lambda}, (\mathfrak{a}_{k+1}^\lambda, b^\lambda), \widetilde{E_{k+1}^\lambda}).$$

where

$$\widetilde{W_{k+1}^\lambda} = \widetilde{W_k^\lambda} \times_{W_k} W_{k+1}, \mathfrak{a}_{k+1}^\lambda = \mathfrak{a}_k^\lambda \mathcal{O}_{\widetilde{W_{k+1}^\lambda}}, \widetilde{E_{k+1}^\lambda} = {\pi_{k+1}^\lambda}^{-1}(\widetilde{E_k^\lambda})$$

and where π_{k+1}^λ *is the projection onto the first factor.*

Take the smooth morphism of basic objects

$$(W_k, (J_k, b), E_k) \overset{\pi_{k+1}}{\longleftarrow} (W_{k+1}, (J_{k+1}, b), E_{k+1}).$$

Then

$$F_{k+1} := \cup \mathrm{Sing}(\mathfrak{a}_{k+1}^\lambda, b^\lambda)$$

is a closed subset of W_{k+1} *with*

$$F_{k+1} \cap W_{k+1}^\lambda = \mathrm{Sing}(\mathfrak{a}_{k+1}^\lambda, b^\lambda),$$

and the extended sequence belongs to $\mathfrak{C}$*, i.e.,* $F_{k+1} = \mathrm{Sing}(J_{k+1}, b)$ *and*

$$(F_0, (W_0, E_0)) \leftarrow \cdots \leftarrow (F_k, (W_k, E_k)) \leftarrow (F_{k+1}, (W_{k+1}, E_{k+1})) \in \mathfrak{C}.$$

(GB-3) There exists c $(= b!) \in \mathbb{N}$ *such that* $c \geq b^\lambda \quad \forall \lambda$.

Remark 3-2.

(i) The main point of the key inductive lemma is that the problem of resolution of singularities for a basic object $(W, (J, b), E) = (W_0, (J_0, b), E_0)$ (under conditions 1 and 2) of dimension d, i.e., the problem of finding a sequence of transformations

$$(W_0, (J_0, b), E_0) \leftarrow \cdots (W_k, (J_k, b), E_k) \text{ with } \mathrm{Sing}(J_k, b) = \emptyset,$$

which gives rise to a sequence of transformations in $\mathfrak{C}$

$$(F_0, (W_0, E_0)) \leftarrow \cdots \leftarrow (F_k, (W_k, E_k)) \text{ with } F_k = \emptyset,$$

can be reduced to that of charts of basic objects $\{(\widetilde{W_0^\lambda}, (\mathfrak{a}_0^\lambda, b^\lambda), \widetilde{E_0^\lambda})\}$ of dimension $d - 1$, if we could find the global centers which are permissible with respect to all the local charts.

That is to say, starting with the trivial sequence (condition (GB-0)), we build up a resolution sequence of a (simple) basic object of dimension d by extending the trivial one via the repeated use of condition (GB-2), based upon the resolution

sequence of the charts of basic objects in dimension $d-1$, which is obtained by induction.

Together with condition (GB-2) which characterizes the sequences in the collection $\mathfrak{C}$ and with condition (GB-3) which ensures the boundedness of the integers b^λ in order to gurantee the descending chain condition of our invariants, (GB-0,1,2,3) will be used as the defining conditions for general basic objects in Definition 4-1.

We discuss the shortcomings of the key inductive lemma toward a complete inductive algorithm in (ii), (iii), and (iv) below.

(ii) (**Simple basic object**) Condition 1 implies (cf. Lemma 1-4) that

$$\nu_p(\Delta^{b-1}(J)) = 1 \quad \forall p \in V(\Delta^{b-1}(J)),$$

which is equivalent to the characterization of what we call a **simple basic object**:

A basic object $(W,(J,b),E)$ is called simple if

$$\nu_p(J) = b = b_{\max} = \max\{\nu_q(J); q \in W\} \quad \forall p \in \operatorname{Sing}(J,b)(= V(\Delta^{b-1}(J))).$$

Conversely, for a simple basic object $(W,(J,b),E)$ it is easy to find an open covering $\{W^\lambda\}$ and smooth hypersurfaces $W_h^\lambda \subset W^\lambda$ with the property $I(W_h^\lambda) \subset \Delta^{b-1}(J)$ and hence satisfying condition 1. (See Remark 1-5 where we discuss the primitive but fundamental idea toward the inductional argument via the notion of a hypersurface of maximal contact.)

Thus the key inductive lemma restrictively applies only to simple basic objects, for which an open covering satisfying condition 1 comes almost for free.

(iii) Even if we start from a simple basic object of dimension d, the resulting basic objects $\{(\widetilde{W_0^\lambda},(\mathfrak{a}_0^\lambda,b^\lambda),\widetilde{E_0^\lambda})\}$ of dimension $(d-1)$, in Case B, are almost never simple. Therefore, even though we call this lemma with the adjective "inductive", it is not clear at this point (until Chapter 5) how this induction would actually work.

(iv) Condition 2 is more problematic, if one tries to see the inductive structure of a possible proof for resolution of singularities in a naive way suggested by the above lemma:

It is NOT true that we can find an open covering $\{W^\lambda\}$ which satisfies conditions 1 and 2 for an arbitrary simple basic object.

Take $(W,(J,b),E) = (\mathbb{A}^2 = \operatorname{Spec}[x,y],(\langle x^2\rangle),2),\{y-x^2=0\})$. It is easy to check that $\nu_p(J) = 2 = b \ \ \forall p \in \operatorname{Sing}(J,b) = \{x=0\}$ and hence that it is a simple basic object. Observe $\Delta^{b-1}(J) = (x)$ and hence by condition 2 the smooth hypersurface W_h^λ has to contain $\{x=0\} \cap W^\lambda$, no matter how we choose an open covering $\{W^\lambda\}$. However, in any open subset (in the covering) containing $(0,0)$ the hypersurface $\{x=0\}$ does not cross $E=\{y-x^2\}$ transversally, failing to satisfy condition 2.

(v) The shortcomings of the key inductive lemma expressed in the above (ii), (iii), and (iv) and the problem of how to find the global centers permissible with respect the local charts expressed in (i), will be so elegantly and beautifully resolved in Chapter 5 via the power of the t-invariant. See also Chapter 6 for a more-down-to-earth interpretation of this inductive procedure.

(vi) Since $E = \{H_1, ..., H_r\}$ is a collection of hypersurfaces and E^λ is a collection of the restrictions of the hypersurfcaes to the open subset W^λ, it is more appropriate logically to write $E^\lambda = \{H_1 \cap W^\lambda, ..., H_r \cap W^\lambda\}$ or $E^\lambda = \{H_1|_{W^\lambda}, ..., H_r|_{W^\lambda}\}$ than to write $E^\lambda = E \cap W^\lambda$ or $E^\lambda = E|_{W^\lambda}$, which we use, however, by abuse of and for simplicity of notation. We also write $\widetilde{E_0^\lambda} = E_0^\lambda \cap \widetilde{W_0^\lambda}$ instead of $\widetilde{E_0^\lambda} = \{H_1 \cap \widetilde{W_0^\lambda}, ..., H_r \cap \widetilde{W_0^\lambda}\}$.

Proof of Lemma 3-1.

Case A: $R(1)(\mathrm{Sing}(J, b)) \neq \emptyset$.

Since condition 1 implies $\mathrm{Sing}(J, b) \cap W^\lambda \subset W_h^\lambda$, we have

$$\begin{aligned} &\text{either } R(1)(\mathrm{Sing}(J, b)) \cap W^\lambda = \emptyset \\ &\text{or } R(1)(\mathrm{Sing}(J, b)) \cap W^\lambda \text{ open and closed in } W_h^\lambda. \end{aligned}$$

Therefore, we conclude that $R(1)(\mathrm{Sing}(J, b))$ is smooth, open and closed in $\mathrm{Sing}(J, b)$ and that it is permissible with respect to E, since W_h^λ is smooth and since W_h^λ is permissible with respect to $E \cap W^\lambda$ for each $\lambda \in \Lambda$.

Take

$$(W, (J, b), E) = (W_0, (J_0, b), E_0) \leftarrow (W_1, (J_1, b), E_1)$$

to be the transformation with center $Y_0 = R(1)(\mathrm{Sing}(J, b))$.

Since Y_0 is a smooth divisor of codimension one, the ambient space remains unchanged, i.e., $W_0 \overset{\sim}{\leftarrow} W_1$. What changes is the ideal, from J_0 to J_1. By definition, we have

$$J_0 \mathcal{O}_{W_1} = I(H_{r+1})^b \cdot J_1$$

and hence

$$\nu_p(J_1) = b - b = 0 < b$$

for any codimension one point p which is the generic point of an irreducible component in $H_{r+1} = Y_0 = R(1)(\mathrm{Sing}(J, b))$. Therefore, $\mathrm{Sing}(J_1, b) \subset \mathrm{Sing}(J, b)$ has no codimension one point, i.e.,

$$R(1)(\mathrm{Sing}(J_1, b)) = \emptyset.$$

Case B: $R(1)(\mathrm{Sing}(J, b)) = \emptyset$.

We take with respect to the given open covering $\{W^\lambda\}$ a collection of basic objects

$$\{(\widetilde{W_0^\lambda}, (\mathfrak{a}^\lambda, b^\lambda), \widetilde{E_0^\lambda}) = (W_h^\lambda, (C(J^\lambda), b!), W_h^\lambda \cap E)\},$$

where $J^\lambda = J|_{W^\lambda}$ and where the **coefficient ideal** $C(J^\lambda)$ is defined to be

$$C(J^\lambda) := \Sigma_{i=0}^{b-1} \Delta^i (J^\lambda)^{\frac{b!}{b-i}} \mathcal{O}_{W_h^\lambda}.$$

(The definition of the coefficient ideal forms the technical core of the proof of the key inductive lemma. For a background motivation, the reader is invited to look at Remark 3-7 regarding the Tschirmhausen transformation.)

Note that in order to verify conditions (GB-0,1,2,3) for $(W,(J,b),E)$ with $\{(\widetilde{W_0^\lambda},(\mathfrak{a}^\lambda,b^\lambda),\widetilde{E_0^\lambda})\}$, it suffices to verify conditions (GB-0,1,2,3) for $(W^\lambda,(J|_{W^\lambda},b),E|_{W^\lambda})$ with $(\widetilde{W_0^\lambda},(\mathfrak{a}^\lambda,b^\lambda),\widetilde{E_0^\lambda})$ for each λ.

Therefore, we drop the superscript λ from the following argument. That is to say, we assume $(W,(J,b),E)$ is a basic object with a smooth hypersurface $W_h \subset W$, embedded as a closed subscheme, such that

1. $I(W_h) \subset \Delta^{b-1}(J)$,
2. W_h is permissible with respect to E, and W_h is not contained in E, i.e., $W_h \not\subset E$.

(Note, however, that we do keep the superscript λ in b^λ, since it may actually be different from b.)

Claim 3-3. *:* $\mathrm{Sing}(J,b) = \mathrm{Sing}(C(J),b!)$.

Proof.

We observe that

$$\begin{aligned}
& p \in \mathrm{Sing}(J,b) \ (\subset W_h) \\
\Longleftrightarrow\ & \nu_p(\Delta^i(J)) \geq b-i \text{ for } i = 0,...,b-1 \\
\Longleftrightarrow\ & \nu_p(\Delta^i(J)^{\frac{b!}{b-i}}) \geq b! \text{ for } i = 0,...,b-1 \\
\Longrightarrow\ & \nu_p(\Sigma_{i=0}^{b-1}\Delta^i(J)^{\frac{b!}{b-i}}\mathcal{O}_{W_h}) \geq b! \\
\Longleftrightarrow\ & p \in \mathrm{Sing}(C(J),b!)
\end{aligned}$$

and that

$$\begin{aligned}
& p \notin \mathrm{Sing}(J,b) \\
\Longleftrightarrow\ & \Delta^i(J)_p = \mathcal{O}_{W,p} \text{ for some } i = 0,...,b-1 \\
\Longrightarrow\ & \Sigma_{i=0}^{b-1}\Delta^i(J)^{\frac{b!}{b-i}}\mathcal{O}_{W_h,p} = \mathcal{O}_{W_h,p} \\
\Longrightarrow\ & p \notin \mathrm{Sing}(C(J),b!).
\end{aligned}$$

This proves the assertion.

Claim 3-4 (Giraud's Lemma). *Let $(W,(J,b),E)$ be a basic object and $W_h \subset W$ be a smooth hypersurface, embedded as a closed subscheme, satisfying conditions 1 and 2 as above.*

Case T: $(W,(J,b),E) \overset{\pi_1}{\leftarrow} (W_1,(J_1,b),E_1)$ is a transformation with permissible center $Y \subset W$ for $(W,(J,b),E)$, i.e., Y is permissible with respect to E and $Y \subset \mathrm{Sing}(J,b)$.

In this case, $(W_h)_1 \subset W_1$, the strict transform of W_h, is a smooth hypersurface satisfying conditions 1 and 2.

Case S: $(W,(J,b),E) \overset{\pi_1}{\leftarrow} (W_1,(J_1,b),E_1)$ is a smooth morphism.

In this case, $(W_h)_1 = \pi_1^{-1}(W_h) \subset W_1$ is a smooth hypersurface satisfying conditions 1 and 2.

Proof.

The assertions in Case S are obvious, since π_1 is étale equivalent to the projection $W \leftarrow W \times \mathbb{A}^n$ for some n. In Case T, since Y is permissible with respect to E and $Y \subset \mathrm{Sing}(J,b) \subset W_h$, where W_h is permissible with respect to E, we see that Y is permissible with respect to $\{H_1, ..., H_r, W_h\}$. Therefore, we conclude that $H_{r+1} = \pi_1^{-1}(Y)$ is permissible with respect to $\{H_1, ..., H_r, (W_h)_1\}$, where H_i denotes the strict transform of H_i (which is denoted by the same letter by abuse of notation) and hence that $(W_h)_1$ is permissible with respect to $E_1 = \{H_1, ..., H_r, H_{r+1}\}$. This shows condition 2 for $(W_h)_1$.

Moreover, Lemma 1-13 implies

$$I((W_h)_1) = \frac{1}{I(H_{r+1})} I(W_h)\mathcal{O}_{W_1} \subset \frac{1}{I(H_{r+1})} \Delta^{b-1}(J) \subset \Delta^{b-1}(J_1),$$

and hence that

$$(W_h)_1 \supset \mathrm{Sing}(J_1, b).$$

This shows condition 1 for $(W_h)_1$.

We go back to the analysis of **Case B**.

Let

$$(W, (J,b), E) = (W_0, (J_0, b), E_0) \leftarrow \cdots \leftarrow (W_k, (J_k, b), E_k)$$

be a sequence of transformations and smooth morphisms of basic objects.

Claim 3-4 implies that there corresponds a sequence of transformations and smooth morphisms of pairs with the natural immersions

$$\begin{array}{ccccc}
(W_0, E_0) & \longleftarrow & \cdots & \longleftarrow & (W_k, E_k) \\
\uparrow & & & & \uparrow \\
(\widetilde{W_0}, \widetilde{E_0}) & \longleftarrow & \cdots & \longleftarrow & (\widetilde{W_k}, \widetilde{E_k})
\end{array}$$

where

$\widetilde{W_0} = (W_0)_h = W_h$, and inductively

$\widetilde{W_{i-1}} = (W_{i-1})_h \subset W_{i-1}$ is a smooth hypersurface satisfying conditions 1 and 2, and

$\widetilde{W_i} = (W_i)_h = ((W_{i-1})_h)_1 \subset W_i$

such that

$$\mathrm{Sing}(J_i, b) \subset \widetilde{W_i} \text{ for } i = 0, 1, ..., k.$$

Claim 3-5. *The following assertions hold inductively for $i = 0, 1, ..., k$:*

(i) $\mathrm{Sing}(J_i, b) = \mathrm{Sing}(C(J_i), b!) = \mathrm{Sing}(C(J_0)_i, b!)$.

(ii) (The assertion (ii) is void when $i = 0$.)

Case T: $(W_{i-1}, (J_{i-1}, b), E_{i-1}) \stackrel{\pi_i}{\leftarrow} (W_i, (J_i, b), E_i)$ *is a transformation with center* $Y_{i-1} \subset \mathrm{Sing}(J_{i-1}, b) \subset W_{i-1}$ *and* $\pi_i^{-1}(Y_{i-1}) = H_i$.

(Note that by induction $Y_{i-1} \subset \mathrm{Sing}(J_{i-1}, b) = \mathrm{Sing}(C(J_0)_{i-1}, b!)$ *and hence that we have an induced transformation of basic objects*

$$(\widetilde{W_{i-1}}, \mathfrak{a}_{i-1} = C(J_0)_{i-1}, \widetilde{E_{i-1}}) \leftarrow (\widetilde{W_i}, \mathfrak{a}_i = C(J_0)_i, \widetilde{E_i}).)$$

The ideal $[\Delta^{b-j}(J_0)]_{i-1}\mathcal{O}_{W_i}$ *is divisible by* $I(H_i)^j$*, and we set*

$$[\Delta^{b-j}(J_0)]_i = \frac{1}{I(H_i)^j}[\Delta^{b-j}(J_0)]_{i-1}\mathcal{O}_{W_i} \textit{ for } j = 1, ..., b$$

and we have an equality

$$C(J_0)_i = \Sigma_{j=1}^{b}[\Delta^{b-j}(J_0)]_i^{\frac{b!}{j}}\mathcal{O}_{\widetilde{W_i}}.$$

Case S: $(W_{i-1}, (J_{i-1}, b), E_{i-1}) \stackrel{\pi_i}{\leftarrow} (W_i, (J_i, b), E_i)$ *is a smooth morphism.*
(We have an induced smooth morphism of basic objects

$$(\widetilde{W_{i-1}}, \mathfrak{a}_{i-1} = C(J_0)_{i-1}, \widetilde{E_{i-1}}) \leftarrow (\widetilde{W_i}, \mathfrak{a}_i = C(J_0)_i, \widetilde{E_i}).)$$

In this case, we set

$$[\Delta^{b-j}(J_0)]_i = [\Delta^{b-j}(J_0)]_{i-1}\mathcal{O}_{W_i} = \frac{1}{I(H_i)^j}[\Delta^{b-j}(J_0)]_{i-1}\mathcal{O}_{W_i} \textit{ for } j = 1, ..., b$$

where we use the convention $H_i = \emptyset$ *and* $I(H_i) = \mathcal{O}_{W_i}$*, and we have an equality*

$$C(J_0)_i = \Sigma_{j=1}^{b}[\Delta^{b-j}(J_0)]_i^{\frac{b!}{j}}\mathcal{O}_{\widetilde{W_i}} = C(J_0)_{i-1}\mathcal{O}_{\widetilde{W_i}}.$$

Moreover, both in Case T and in Case S, we have an inclusion

$$[\Delta^{b-j}(J_0)]_i \subset \Delta^{b-j}(J_i) \textit{ for } j = 1, ..., b.$$

We understand by definition

$$[\Delta^{b-j}(J_0)]_0 = \Delta^{b-j}(J_0) \textit{ and } C(J_0)_0 = C(J_0).$$

(iii) At any closed point $\xi_i \in \mathrm{Sing}(\mathfrak{a}_i, b^\lambda) = \mathrm{Sing}(C(J_0)_i, b!)$*, considered as a closed point in* W_i *(*$\xi_i \in \mathrm{Sing}(\mathfrak{a}_i, b^\lambda) \subset \widetilde{W_i} \subset W_i$*), there exists a system of regular parameters* $z_i, x_{i,1}, ..., x_{i,d-1}$*, where* $d = \dim W_i$*, such that*

(a) $I(\widetilde{W_i}) = \langle z_i \rangle$*,*

(b) setting $R_i = \widehat{\mathcal{O}_{W_i,\xi_i}}, \overline{R_i} = \widehat{\mathcal{O}_{\widetilde{W_i},\xi_i}}$ *there is a set of generators* $\{f_i^{(\sigma)}\}$ *for the ideal* J_iR_i

$$f_i^{(\sigma)} = \Sigma_\alpha a_{i,\alpha}^{(\sigma)} z_i^\alpha \quad \textit{where} \quad a_{i,\alpha}^{(\sigma)} \in k[[x_{i,1}, ..., x_{i,d-1}]] \subset k[[z_i, x_{i,1}, ..., x_{i,d-1}]] = R_i$$

so that

$$(a_{i,\alpha}^{(\sigma)})^{\frac{b!}{b-\alpha}} \in C(J_0)_i\overline{R_i} \textit{ for all } \alpha \textit{ with } \alpha < b.$$

Remark 3-6.

The claim may look technical at first sight. The calamity is that the operation of taking extensions and that of taking transformations of an ideal (i.e., taking the pull-back divided by some suitable multiple of the ideal defining the exceptional divisor) do not commute under a transformation of basic objects $(W_0,(J_0,b),E) \stackrel{\pi_1}{\leftarrow} (W_1,(J_1,b),E_1)$, that is to say, in general

$$\frac{1}{I(H_{r+1})^j}\Delta^{b-j}(J_0)\mathcal{O}_{W_1} \neq \Delta^{b-j}(J_1),$$

where $I(H_{r+1})$ is the ideal defining the exceptional divisor H_{r+1} for π_1. (If the equality were to hold, we would have $C(J_0)_1 = C(J_1)$, from which the assertions in **Case B** would have followed directly.) The purpose of the claim is to overcome this calamity using Lemma 1-13, which analyzes the behavior of extensions under transformations.

Proof of Claim 3-5.

(i) The assertion (i) is an immediate consequence of the assertions (ii) and (iii) as follows:

Observe first that Claim 3-3 and Claim 3-4 imply

$$\mathrm{Sing}(J_i,b) = \mathrm{Sing}(C(J_i),b!).$$

The "moreover" part of the assertion (ii) implies $C(J_0)_i \subset C(J_i)$ (Note that the inclusion obviously holds even when $i=0$.) and hence

$$\mathrm{Sing}(C(J_i),b!) \subset \mathrm{Sing}(C(J_0)_i,b!).$$

On the other hand, the assertion (iii) implies that for $p \in \mathrm{Sing}(C(J_0)_i,b!)$ we have

$$\nu_p(a^{(\sigma)}_{i,\alpha}) \geq b-\alpha \text{ and hence } \nu_p(f^{(\sigma)}_i) \geq b,$$

and therefore we have

$$\mathrm{Sing}(C(J_0)_i,b!) \subset \mathrm{Sing}(J_i,b),$$

completing the argument for the assertion (i).

We prove the assertions (ii) and (iii) by induction on i.

For $i=0$, the assertion (iii) is obvious from the definition, while the assertion (ii) is void. (Note that $a^{(\sigma)}_{0,\alpha} = \frac{1}{\alpha!}(\frac{\partial}{\partial z_0})^\alpha f^{(\sigma)}_0|_{z_0=0} \in \Delta^\alpha(J_0)\overline{R_0}$ and hence $(a^{(\sigma)}_{0,\alpha})^{\frac{b!}{b-\alpha}} \in C(J_0)\overline{R_0}$.)

Assuming that the assertions hold for $i \leq s$, we prove the assertions for $i = s+1$ by induction.

(ii) In Case S, since π_{s+1} is étale equivalent to the projection $W_s \leftarrow W_s \times \mathbb{A}^n$ for some n, all the assertions immediately hold by induction. So we will concentrate our consideration on Case T where $(W_s,(J_s,b),E_s) \stackrel{\pi_{s+1}}{\leftarrow} (W_{s+1},(J_{s+1},b),E_{s+1})$ is a transformation with center $Y_s \subset W_s$ and $H_{s+1} = \pi^{-1}_{s+1}(Y_s)$.

By the inductional hypothesis, we have

$$[\Delta^{b-j}(J_0)]_s \subset \Delta^{b-j}(J_s) \text{ for } j = 1, ..., b.$$

Lemma 1-13 implies that $\Delta^{b-j}(J_s)\mathcal{O}_{W_{s+1}}$ is divisible by $I(H_{s+1})^j$ and that

$$\frac{1}{I(H_{s+1})^j}\Delta^{b-j}(J_s)\mathcal{O}_{W_{s+1}} \subset \Delta^{b-j}(J_{s+1}).$$

Therefore, we conclude that $[\Delta^{b-j}(J_0)]_s\mathcal{O}_{W_{s+1}}$ is also divisible by $I(H_{s+1})^j$ and that

$$\begin{aligned}[\Delta^{b-j}(J_0)]_{s+1} &= \frac{1}{I(H_{s+1})^j}[\Delta^{b-j}(J_0)]_s \\ &\subset \frac{1}{I(H_{s+1})^j}\Delta^{b-j}(J_s) \subset \Delta^{b-j}(J_{s+1}).\end{aligned}$$

(iii) Let $\xi_{s+1} \in \mathrm{Sing}(C(J_0)_{s+1}, b!) \subset \widetilde{W_{s+1}}$ be a closed point and $\xi_s = \pi_{s+1}(\xi_s)$ be its image in $\widetilde{W_s} \subset W_s$.

We may only consider the case where $\xi_{s+1} \in H_{s+1}$, since otherwise the assertions are automatic by the inductional hypothesis as π_{s+1} is isomorphic in a neighborhood of ξ_{s+1}.

By taking a change of variables involving only $x_{s,l}$'s ($l = 1, ..., c = \dim_{\xi_s} Y_s$) in the system of regular parameters for R_s given by the inductional hypothesis for (iii) where Y_s is defined by the ideal $\langle z_s, x_{s,1}, ..., x_{s,c-1}\rangle$, we may assume that $R_{s+1} = \widehat{\mathcal{O}_{W_{s+1},\xi_{s+1}}}$ has a system of regular parameters $z_{s+1}, x_{s+1,1}, ..., x_{s+1,c-1}, x_{s+1,c}, ..., x_{s+1,d-1}$ with $d = \dim W_{s+1} = \dim W_s$ such that

$$\begin{aligned} I(H_{s+1})_{\xi_{s+1}} &= \langle x_{s+1,1}\rangle \\ x_{s+1,1} &= x_{s,1} \\ I(\widetilde{W_{s+1}})_{\xi_{s+1}} &= \langle z_{s+1}\rangle \\ z_{s+1} &= \frac{z_s}{x_{s,1}} \\ x_{s+1,l} &= \frac{x_{s,l}}{x_{s,1}} \text{ for } l = 2, ..., c-1 \\ x_{s+1,l} &= x_{s,l} \text{ for } l = c, ..., d-1. \end{aligned}$$

Take a set of generators $\{f_{s+1}^{(\sigma)}\}$ for $J_{s+1}R_{s+1}$ where

$$f_{s+1}^{(\sigma)} = \frac{f_s^{(\sigma)}}{x_{s,1}^b} = \Sigma_\alpha a_{s+1,\alpha}^{(\sigma)} z_{s+1}^\alpha$$

so that

$$a_{s+1,\alpha}^{(\sigma)} = \frac{a_{s,\alpha}^{(\sigma)}}{x_{s,1}^{b-\alpha}}.$$

Therefore, we conclude

$$
\begin{aligned}
(a_{s+1,\alpha}^{(\sigma)})^{\frac{b!}{b-\alpha}} &= (\frac{a_{s,\alpha}^{(\sigma)}}{x_{s,1}^{b-\alpha}})^{\frac{b!}{b-\alpha}} \\
&= (a_{s,\alpha}^{(\sigma)})^{\frac{b!}{b-\alpha}} \cdot (\frac{1}{x_{s,1}})^{b!} \\
&\in C(J_0)_s \overline{R_s} \cdot (\frac{1}{x_{s,1}})^{b!} \cdot \overline{R_{s+1}} \text{ (by inductional hypothesis)} \\
&= \Sigma_{j=1}^{b} [\Delta^{b-j}(J_0)]_s^{\frac{b!}{j}} \mathcal{O}_{\widetilde{W_s}} \cdot \overline{R_s} \cdot (\frac{1}{x_{s,1}})^{b!} \cdot \overline{R_{s+1}} \\
&= \Sigma_{j=1}^{b} \{ \frac{1}{I(H_{s+1})^j} [\Delta^{b-j}(J_0)]_s \}^{\frac{b!}{j}} \mathcal{O}_{\widetilde{W_{s+1}}} \cdot \overline{R_{s+1}} \\
&= C(J_0)_{s+1} \overline{R_{s+1}}.
\end{aligned}
$$

This completes the proof of Claim 3-5.

We go back to the proof of Lemma 3-1.

Condition (GB-0) is obvious. Condition (GB-1) is an immediate consequence of Claim 3-5 (i), and so is condition (GB-2). Condition (GB-3) is obvious from the construction.

This completes the proof of Lemma 3-1 (the key inductive lemma).

Remark 3-7.

(i) Consider a basic object $(W, (J, b), E)$ where

$$
\begin{cases}
W = \mathbb{A}^d = \text{Spec } k[x_1, ..., x_{d-1}, x_d] \\
J = \langle f \rangle \text{ with} \\
f = x_d^n + c_{n-2} x_d^{n-2} + \cdots + c_1 x_d + c_0 \text{ with } c_i \in k[x_1, ..., x_{d-1}] \\
b = n \\
E = \emptyset.
\end{cases}
$$

We would like to emphasize that f is already in the form after a **Tschirnhausen transformation**, which can be always carried out over a field of characteristic zero, i.e.,

the coefficient c_{n-1} of the term x_d^{n-1} is equal to 0 in f.

This implies that (Recall $b = n$.)

$$\frac{1}{(b-1)!} \frac{\partial^{b-1} f}{\partial x_d^{b-1}} = x_d \in \Delta^{b-1}(J).$$

Set

$$W_h = V(x_d).$$

Then conditions 1 and 2 of Lemma 3-1 are clearly satisfied.

Observe

$$\begin{aligned} p \in \mathrm{Sing}(J,b) &\Longleftrightarrow x_d(p) = 0 \quad \& \quad \nu_p(c_i) \geq b - i \text{ for } i = 0, ..., b-1 \\ &\Longleftrightarrow x_d(p) = 0 \quad \& \quad \text{ for } i = 0, ..., b-1 \\ &\qquad \nu_p(\langle c_i \rangle) \geq b - i, \nu_p(\Delta^1 \langle c_{i-1} \rangle)) \geq b - i, ..., \nu_p(\Delta^i \langle c_0 \rangle) \geq b - i \\ &\Longleftrightarrow p \in W_h \quad \& \quad \nu_p(\Sigma_{i=0}^{b-1} \{\langle c_i \rangle + \Delta^1 \langle c_{i-1} \rangle + \cdots + \Delta^i \langle c_0 \rangle\}^{\frac{b!}{b-i}}) \geq b!. \end{aligned}$$

This observation might give a justification for calling

$$\Sigma_{i=0}^{b-1} \Delta^i(J)^{\frac{b!}{b-i}} \mathcal{O}_{W_h} = \Sigma_{i=0}^{b-1} \{\langle c_i \rangle + \Delta^1 \langle c_{i-1} \rangle + \cdots + \Delta^i \langle c_0 \rangle\}^{\frac{b!}{b-i}}$$

the coefficient ideal.

(ii) Let $(W,(J,b,E)$ be a (simple) basic object with a smooth hypersurface $W_h \subset W$ satisfying conditions 1 and 2:
1. $I(W_h) \subset \Delta^{b-1}(J)$,
2. W_h is permissible with respect to E and $W_h \not\subset E$.

Since

$$\mathrm{Sing}(J,b) = V(\Delta^{b-1}(J)) = V(\Delta^{b-1}(J))|_{W_h},$$

it might look plausible (and simpler) to consider the basic object

$$(W_h, (D(J), 1), E \cap W_h) \text{ with } D(J) = \Delta^{b-1}(J)\mathcal{O}_{W_h},$$

instead of

$$(W_h, (C(J), b!), E \cap W_h) \text{ with } C(J) = \Sigma_{i=0}^{b-1} \Delta^i(J)\mathcal{O}_{W_h},$$

as a candidate for the basic object of dimension one less in order for the key inductive lemma to work.

This alternative definition, however, does NOT work.

Look at a simple basic object $(W,(J,b),E)$ where

$$\begin{cases} W = \mathbb{A}^2 = \mathrm{Spec}\ k[x,y] \\ J = \langle x^2 - y^3 \rangle \\ b = 2 \\ E = \emptyset \end{cases}$$

with a smooth hypersurface $W_h = \{x = 0\}$, clearly satisfying conditions 1 and 2 of Lemma 3-1.

If we take the transformation of basic objects

$$(W,(J,b),E) = (W_0,(J_0,b),E_0) \leftarrow (W_1,(J_1,b),E_1),$$

which is the blowup of the origin, then in the chart with coordinate system $(t = \frac{x}{y}, y)$ we have $J_1 = \langle t^2 - y \rangle$ and hence

$$\mathrm{Sing}(J_1,b) = \mathrm{Sing}(J_1,b)|_{(W_h)_1} = \emptyset.$$

On the other hand, if we look at the corresponding transformation of basic objects

$$\begin{aligned}(W_h, (D(J) = \Delta^{b-1}(J)\mathcal{O}_{W_h}, 1), E \cap W_h) &= ((W_h)_0, (D(J)_0, b), (E_h)_0) \\ &\leftarrow ((W_h)_1, (D(J)_1, 1), (E_h)_1),\end{aligned}$$

then we have $W_h = (W_h)_0 = (W_h)_1 = \mathbb{A}^1 = \text{Spec } k[y]$ with $D(J)_0 = \langle y^2 \rangle$ and $D(J)_1 = \langle y \rangle$, and hence

$$\text{Sing}(D(J)_1, 1) \neq \emptyset.$$

Therefore, we have

$$\text{Sing}(J_1, b) \neq \text{Sing}(D(J)_1, 1),$$

failing to have the desired equality between the singular loci.

It is worthwhile to note that if the equality in Remark 3-6, which was remarked there not to hold, were true, then the above simpler candidate would work.

It is our intention to emphasize the subtlety involving the definition of the coefficient ideal and Claim 3-5.

CHAPTER 4. GENERAL BASIC OBJECTS AND INVARIANTS

In this chapter, we introduce the notion of a general basic object, which turns out to be the right framework, in the solution by Encinas and Villamayor, to extract the inductive nature of the problem of resolution of singularities. We also show that the invariants defined on the singular loci of the individual basic objects, as in Definition 1-10, in the charts of a general basic object, patch together to define well-defined invariants on the singular locus of the general basic object.

Definition 4-1 (General basic object). *A general basic object over* $(F_0, (W_0, E_0))$, *where* (W_0, E_0) *is a pair (cf. Definition 1-6) and* $F_0 \subset W_0$ *is a closed subset, with a d-dimensional structure* ($d \leq \dim W_0$), *is a collection* $\mathfrak{C}$ *of sequences of transformations and smooth morphisms of pairs with specified closed subsets starting with* $(F_0, (W_0, E_0))$

$$(F_0, (W_0, E_0)) \leftarrow \cdots \leftarrow (F_k, (W_k, E_k))$$

and an open covering $\{W_0^\lambda\}_{\lambda \in \Lambda}$ *with the following data* $\mathcal{D}_\lambda$ *for each* $\lambda \in \Lambda$*:*

(i) $j_0^\lambda : (\widetilde{W_0^\lambda}, \widetilde{E_0^\lambda}) \hookrightarrow (W_0^\lambda, E_0^\lambda)$ *is an immersion of pairs where* $\dim \widetilde{W_0^\lambda} = d$, *that is to say,* $\widetilde{W_0^\lambda} \hookrightarrow W_0^\lambda$ *is a closed immersion of a d-dimensional smooth variety* $\widetilde{W_0^\lambda}$ *into* W_0^λ, $\widetilde{W_0^\lambda}$ *is permissible with respect to* $E_0^\lambda = E_0 \cap W_0^\lambda$ *and* $\widetilde{W_0^\lambda} \not\subset E_0^\lambda$, *and* $\widetilde{E_0^\lambda} = E_0^\lambda \cap \widetilde{W_0^\lambda}$,

(ii) a basic object $(\widetilde{W_0^\lambda}, (\mathfrak{a}_0^\lambda, b^\lambda), \widetilde{E_0^\lambda})$,

satisfying the following conditions (GB-0,1,2,3):

(GB-0) The trivial sequence consisting only of $(F_0, (W_0, E_0))$ *is in the collection* $\mathfrak{C}$, *i.e.,*

$$(F_0, (W_0, E_0)) \in \mathfrak{C}$$

and

$$F_0 = \cup \mathrm{Sing}(\mathfrak{a}_0^\lambda, b^\lambda) \text{ with } F_0 \cap W_0^\lambda = \mathrm{Sing}(\mathfrak{a}_0^\lambda, b^\lambda).$$

(GB-1) With any sequence of transformations and smooth morphisms in the collection $\mathfrak{C}$

$$(F_0, (W_0, E_0)) \leftarrow \cdots \leftarrow (F_k, (W_k, E_k))$$

there corresponds for each λ *a sequence of transformations (with the same centers) and (the same) smooth morphisms (obtained by taking the Cartesian products)*

$$(\widetilde{W_0^\lambda}, (\mathfrak{a}_0^\lambda, b^\lambda), \widetilde{E_0^\lambda}) \leftarrow \cdots \leftarrow (\widetilde{W_k^\lambda}, (\mathfrak{a}_k^\lambda, b^\lambda), \widetilde{E_k^\lambda})$$

with the natural immersions

$$\begin{array}{ccccc}
(W_0^\lambda, E_0^\lambda) & \longleftarrow & \cdots & \longleftarrow & (W_k^\lambda, E_k^\lambda) \\
\uparrow & & & & \uparrow \\
(\widetilde{W_0^\lambda}, \widetilde{E_0^\lambda}) & \longleftarrow & \cdots & \longleftarrow & (\widetilde{W_k^\lambda}, \widetilde{E_k^\lambda})
\end{array}$$

and we have

$$F_i = \cup \mathrm{Sing}(\mathfrak{a}_i^\lambda, b^\lambda) \text{ with } F_i \cap W_i^\lambda = \mathrm{Sing}(\mathfrak{a}_i^\lambda, b^\lambda)$$

for $i = 0, 1, ..., k$.

(We note here that in the above clause "there corresponds ...", it is required that whenever $(W_{i-1}, E_{i-1}) \overset{\pi_i}{\leftarrow} (W_i, E_i)$ *is a transformation with center* $Y_{i-1} \subset W_{i-1}$, *the center* Y_{i-1} *is permissible for* $(\widetilde{W_{i-1}^\lambda}, (\mathfrak{a}_{i-1}^\lambda, b^\lambda), \widetilde{E_{i-1}^\lambda})$, *i.e.,* $Y_{i-1} \cap W_{i-1}^\lambda \subset \widetilde{W_{i-1}^\lambda}$, $Y_{i-1} \cap W_{i-1}^\lambda$ *is permissible with respect to* $\widetilde{E_{i-1}^\lambda}$, *and* $Y_{i-1} \cap W_{i-1}^\lambda \subset \mathrm{Sing}(\mathfrak{a}_{i-1}^\lambda, b^\lambda)$*.)*

(GB-2) Let

$$(F_0, (W_0, E_0)) \leftarrow \cdots \leftarrow (F_k, (W_k, E_k))$$

be a sequence of transformations and smooth morphisms in $\mathfrak{C}$ *and*

$$\{(\widetilde{W_0^\lambda}, (\mathfrak{a}_0^\lambda, b^\lambda), \widetilde{E_0^\lambda}) \leftarrow \cdots \leftarrow (\widetilde{W_k^\lambda}, (\mathfrak{a}_k^\lambda, b^\lambda), \widetilde{E_k^\lambda})\}$$

the corresponding sequences (indexed by $\lambda \in \Lambda$*) of transformations and smooth morphisms as in (GB-1).*

We take a morphism of pairs $(W_k, E_k) \overset{\pi_{k+1}}{\leftarrow} (W_{k+1}, E_{k+1})$ *which is either in Case T or Case S.*

Case T: $(W_k, E_k) \overset{\pi_{k+1}}{\leftarrow} (W_{k+1}, E_{k+1})$ *is a transformation with center* $Y_k \subset W_k$, *satisfying the condition that* Y_k *is permissible for each* $(\widetilde{W_k^\lambda}, (\mathfrak{a}_k^\lambda, b^\lambda), \widetilde{E_k^\lambda})$, *i.e.,* $Y_k \cap W_k^\lambda \subset \widetilde{W_k^\lambda}$, $Y_k \cap W_k^\lambda$ *is permissible with respect to* $\widetilde{E_k^\lambda}$, *and* $Y_k \cap W_k^\lambda \subset \mathrm{Sing}(\mathfrak{a}_k^\lambda, b^\lambda)$.

Case S: $(W_k, E_k) \overset{\pi_{k+1}}{\leftarrow} (W_{k+1}, E_{k+1})$ *is a smooth morphism.*

Then we have the following assertions on the extension of the original sequence:

Case T: Take for each λ *the corresponding transformation of basic objects with center* $Y_k \cap W_k^\lambda$

$$(\widetilde{W_k^\lambda}, (\mathfrak{a}_k^\lambda, b^\lambda), \widetilde{E_k^\lambda}) \overset{\pi_{k+1}^\lambda}{\leftarrow} (\widetilde{W_{k+1}^\lambda}, (\mathfrak{a}_{k+1}^\lambda, b^\lambda), \widetilde{E_{k+1}^\lambda}).$$

Then

$$F_{k+1} := \cup \mathrm{Sing}(\mathfrak{a}_{k+1}^\lambda, b^\lambda)$$

is a closed subset of W_{k+1} *with*

$$F_{k+1} \cap W_{k+1}^\lambda = \mathrm{Sing}(\mathfrak{a}_{k+1}^\lambda, b^\lambda),$$

and the extended sequence belongs to $\mathfrak{C}$, *i.e.,*

$$(F_0, (W_0, E_0)) \leftarrow \cdots \leftarrow (F_k, (W_k, E_k)) \leftarrow (F_{k+1}, (W_{k+1}, E_{k+1})) \in \mathfrak{C}.$$

Case S: Take for each λ *the corresponding morphism of basic objects*

$$(\widetilde{W_k^\lambda}, (\mathfrak{a}_k^\lambda, b^\lambda), \widetilde{E_k^\lambda}) \overset{\pi_{k+1}^\lambda}{\leftarrow} (\widetilde{W_{k+1}^\lambda}, (\mathfrak{a}_{k+1}^\lambda, b^\lambda), \widetilde{E_{k+1}^\lambda})$$

where

$$\widetilde{W_{k+1}^{\lambda}} = \widetilde{W_k^{\lambda}} \times_{W_k} W_{k+1}, \mathfrak{a}_{k+1}^{\lambda} = \mathfrak{a}_k^{\lambda}\mathcal{O}_{\widetilde{W_{k+1}}}, \widetilde{E_{k+1}^{\lambda}} = {\pi_{k+1}^{\lambda}}^{-1}(\widetilde{E_k^{\lambda}})$$

and where π_{k+1}^{λ} is the projection onto the first factor.

Then

$$F_{k+1} := \cup \mathrm{Sing}(\mathfrak{a}_{k+1}^{\lambda}, b^{\lambda})$$

is a closed subset of W_{k+1} with

$$F_{k+1} \cap W_{k+1}^{\lambda} = \mathrm{Sing}(\mathfrak{a}_{k+1}^{\lambda}, b^{\lambda}),$$

and the extended sequence belongs to $\mathfrak{C}$, i.e.,

$$(F_0, (W_0, E_0)) \leftarrow \cdots \leftarrow (F_k, (W_k, E_k)) \leftarrow (F_{k+1}, (W_{k+1}, E_{k+1})) \in \mathfrak{C}.$$

(GB-3) There exists $c \in \mathbb{N}$ such that $c \geq b^{\lambda} \quad \forall \lambda$.

We denote by $(\mathcal{F}_0, (W_0, E_0))$ a general basic object over $(F_0, (W_0, E_0))$.

We say that the d-dimensional structure of $(\mathcal{F}_0, (W_0, E_0))$ is given by the charts $\{(\widetilde{W_0^{\lambda}}, (\mathfrak{a}_0^{\lambda}, b^{\lambda}), \widetilde{E_0^{\lambda}})\}$ of basic objects of dimension d.

We also say by abuse of language that the collection $\mathfrak{C}$ is represented by the general basic object $(\mathcal{F}_0, (W_0, E_0))$.

We identify two general basic objects $(\mathcal{F}_0, (W_0, E_0))$ and $(\mathcal{F}_0', (W_0, E_0))$ if and only if the collections $\mathfrak{C}$ and $\mathfrak{C}'$, represented by the general basic objects, coincide.

Remark 4-2.

(i) In the previous chapters, the letter d was used to denote the dimension of the ambient space W of a basic object $(W, (J, b), E)$. When we say a general basic object $(\mathcal{F}_0, (W_0, E_0))$ with a d-dimensional structure, the letter d refers to the dimension of the basic objects $\{(\widetilde{W_0}, (\mathfrak{a}_{\lambda}, b^{\lambda}), \widetilde{E_0^{\lambda}}\}$ in the charts, and not to the dimension of W_0. In general, $d \leq \dim W_0$.

(ii) (**The general basic object defined by a basic object**) Let $(W, (J, b), E)$ be a basic object with $d = \dim W$. Set $(F_0, (W_0, E_0)) = (\mathrm{Sing}(J, b), (W, E))$ and take $\mathfrak{C}$ to be the collection of all the sequences of transformations and smooth morphisms

$$(F_0, (W_0, E_0)) \leftarrow \cdots \leftarrow (F_r, (W_r, E_r))$$

induced by the sequences of transformations and smooth morphisms of basic objects

$$(W, (J, b), E) = (W_0, (J_0, b), E_0) \leftarrow \cdots \leftarrow (W_k, (J_k, b), E_k)$$

with

$$F_i = \mathrm{Sing}(J_i, b) \text{ for } i = 0, 1, ..., k.$$

This defines a general basic object over $(F_0, (W_0, E_0))$ with a d-dimensional structure.

Note that two different basic objects, e.g. $(W,(J,b),E)$ and $(W,(J^2,2b),E)$, may define the same general basic object, as they represent the same collection $\mathfrak{C}$.

(iii) (**The meaning of the key inductive lemma**) Let $(W,(J,b),E)$ be a simple basic object with an open covering $\{W^\lambda\}_{\lambda\in\Lambda}$ satisfying conditions 1 and 2 of the key inductive lemma (Lemma 3-1). Then, in case $R(1)(\mathrm{Sing}(J,b)) = \emptyset$, the general basic object $(\mathcal{F}_0,(W_0,E_0))$ with a d-dimensional structure defined as above by the basic object $(W,(J,b),E)$, also has a $(d-1)$-dimensional structure given by $\{(\widetilde{W_0^\lambda},(\mathfrak{a}_0^\lambda,b^\lambda),\widetilde{E_0^\lambda}) = (W_h^\lambda,(C(J^\lambda),b!),E_h^\lambda)\}$.

This drop in the dimension of the structure is the very essence of the key inductive lemma in terms of the notion of general basic objects.

(iv) (**Why smooth morphisms**?) In the process of resolution of singularities, we only consider a sequence of transformations. So why do we have to consider smooth morphisms in the definition of a general basic object ? One main reason is that we would like to use Hironaka's trick (cf. the proof of Definition-Proposition 4-5) to guarantee that the invariants defined on the individual charts patch together to give well-defined invariants on the general basic object. Another reason is that, by including open immersions, we also want to guarantee that the general basic objects behave well under localization.

(v) In the original papers by Encinas and Villamayor, they do not include general smooth morphisms in the sequences to consider for the collection to characterize a general basic object, but include only special smooth morphisms, namely, the projections of the form $W_{i-1} \leftarrow W_i = W_{i-1} \times \mathbb{A}^1$, which they call the restrictions. However, their definition causes a few problems:

$\circ$ It is not clear by their definition whether their general basic objects behave well under localization. That is to say, it is not clear a priori whether the charts $\{(\widetilde{W_0^\lambda} \cap V,(\mathfrak{a}_0^\lambda|_V,b^\lambda),\widetilde{E_0^\lambda} \cap V)\}$ would define a general basic object for an open subset $V \subset W_0$, since the permissiblity of the centers is a global condition. (Though this can be proved using, e.g., the embedded resolution of singularities of the closure of the center taken in the open subset without affecting the open subset itself.)

$\circ$ When we want to discuss the stability of the process of resolution of singularities under smooth morphisms, we would like to have the definition of a smooth morphism between general basic objects, which we would lack under their definition.

Including (general) smooth morphisms into the definition of a general basic object requires no change in the structure of the argument and brings some theoretical clarity avoiding the problems as above.

(vi) (**An alternative way of defining the notion of a general basic object**) Recall that a differentiable (resp. complex) manifold W is defined to consist of a topological space W and an open covering $\{W^\lambda\}_{\lambda\in\Lambda}$ with charts $h^\lambda : W^\lambda \to U^\lambda \subset \mathbb{R}^n$ (resp. $\subset \mathbb{C}^n$) so that the W^λ patch up in the sense that $h^\mu \circ {h^\lambda}^{-1}$ are invertible C^∞-functions (resp. holomorphic functions).

We can give an alternative definition of a general basic object in a similar manner:

A general basic object consists of a pair (W,E) and an open covering $\{W^\lambda\}_{\lambda\in\Lambda}$ with charts $(\widetilde{W^\lambda},(\mathfrak{a}^\lambda,b^\lambda),\widetilde{E^\lambda})$, i.e., basic objects where $\widetilde{W^\lambda} \hookrightarrow W^\lambda$ is a closed

immersion of a d-dimensional smooth variety $\widetilde{W^\lambda}$ into W^λ, $\widetilde{W^\lambda}$ is permissible with respect to $E^\lambda = E \cap W^\lambda$, $\widetilde{W^\lambda} \not\subset E^\lambda$, and where $\widetilde{E^\lambda} = E^\lambda \cap \widetilde{W^\lambda}$. We require that there exists $c \in \mathbb{N}$ such that $c \geq b^\lambda \quad \forall \lambda$.

We also require the following patching condition among the charts.

Let

$$(W^\lambda \cap W^\mu, E \cap W^\lambda \cap W^\mu) = (W_0^{\lambda\mu}, E_0^{\lambda\mu}) \leftarrow \cdots \leftarrow (W_k^{\lambda\mu}, E_k^{\lambda\mu})$$

be a sequence of transformations and smooth morphisms of pairs, starting with the intersection of (W^λ, E^λ) and (W^μ, E^μ), such that there corresponds a sequence of transformations (with the same centers) and (the same) smooth morphisms (obtained by taking the Cartesian products) of basic objects

$$(\widetilde{W^\lambda}, (\mathfrak{a}^\lambda, b^\lambda), \widetilde{E^\lambda}) \cap W^\mu = (\widetilde{W_0^{\lambda\mu}}, (\mathfrak{a}_0^\lambda, b^\lambda), \widetilde{E_0^{\lambda\mu}}) \leftarrow \cdots \leftarrow (\widetilde{W_k^{\lambda\mu}}, (\mathfrak{a}_k^\lambda, b^\lambda), \widetilde{E_k^{\lambda\mu}}).$$

Then there should correspond a sequence of transformations (with the same centers) and (the same) smooth morphisms (obtained by taking the Cartesian products) of basic objects

$$(\widetilde{W^\mu}, (\mathfrak{a}^\mu, b^\mu), \widetilde{E^\mu}) \cap W^\lambda = (\widetilde{W_0^{\mu\lambda}}, (\mathfrak{a}_0^\mu, b^\mu), \widetilde{E_0^{\mu\lambda}}) \leftarrow \cdots \leftarrow (\widetilde{W_k^{\mu\lambda}}, (\mathfrak{a}_k^\mu, b^\mu), \widetilde{E_k^{\mu\lambda}})$$

satisfying

$$\operatorname{Sing}(\mathfrak{a}_i^\lambda, b^\lambda) = \operatorname{Sing}(\mathfrak{a}_i^\mu, b^\mu).$$

The equivalence of this alternative definition and Definition 4-1 is straightforward and its verification is left to the reader as an exercise.

Note 4-3 (Sequence of transformations and smooth morphisms of general basic objects).

Let $(\mathcal{F}_0, (W_0, E_0))$ be a general basic object with a d-dimensional structure given by the charts $\{(\widetilde{W_0^\lambda}, (\mathfrak{a}_0^\lambda, b^\lambda), \widetilde{E_0^\lambda})\}$.

Let

$$(F_0, (W_0, E_0)) \leftarrow \cdots \leftarrow (F_k, (W_k, E_k))$$

be a sequence of transformations and smooth morphisms in $\mathfrak{C}$.

Then the seqeunce induces a general basic object over $(F_i, (W_i, E_i))$ for $i = 0, 1, ..., k$, with a $\{\dim W_i - (\dim W_0 - d)\}$-dimensional structure, in the following way:

We take $\mathfrak{C}_i$ to be the collection of those sequences which are the truncations of the sequences in $\mathfrak{C}$ whose first $(i+1)$-terms coincide with

$$(F_0, (W_0, E_0)) \leftarrow \cdots \leftarrow (F_i, (W_i, E_i))$$

of the given sequence.

We take the data $\mathcal{D}_i^\lambda$ for each λ to consist of:

(i) the induced immersion $j_i^\lambda : (\widetilde{W_i^\lambda}, \widetilde{E_i^\lambda}) \hookrightarrow (W_i^\lambda, E_i^\lambda)$, where $\dim \widetilde{W_i^\lambda} = \dim W_i - (\dim W_0 - d)$,

(ii) the induced basic object $(\widetilde{W_i^\lambda}, (\mathfrak{a}_i^\lambda, b^\lambda), \widetilde{E_i^\lambda})$.

We denote the general basic object as above $(\mathcal{F}_i, (W_i, E_i))$.

Therefore, by abuse of notation, we write the sequence

$$(\mathcal{F}_0, (W_0, E_0)) \leftarrow \cdots \leftarrow (\mathcal{F}_k, (W_k, E_k))$$

and call it a sequence of transformations and smooth morphisms of general basic objects.

Definition 4-4 (Resolution of singularities of a general basic object). *We call a sequence of transformations only of general basic objects*

$$(\mathcal{F}_0, (W_0, E_0)) \leftarrow \cdots \leftarrow (\mathcal{F}_k, (W_k, E_k))$$

resolution of singularities of a general basic object $(\mathcal{F}_0, (W_0, E_0))$ *if*

$$F_k = \emptyset.$$

Definition-Proposition 4-5 (Key invariants of general basic objects). *Let*

$$(\mathcal{F}_0, (W_0, E_0)) \leftarrow \cdots \leftarrow (\mathcal{F}_k, (W_k, E_k))$$

be a sequence of transformations and smooth morphisms of general basic objecs (cf. Note 4-3).

(i) The invariant $\mathrm{ord}_k : F_k \to \frac{1}{c!}\mathbb{Z}\geq 0$ *is a function defined over* F_k *such that for each* λ *it restricts to the invariant* ord_k^λ *of the basic object* $(\widetilde{W_k^\lambda}, (\mathfrak{a}_k^\lambda, b^\lambda), \widetilde{E_k^\lambda})$*, i.e., we have the following commutative diagram*

$$\begin{array}{ccccc} F_k \cap W_k^\lambda & \hookrightarrow & F_k & \stackrel{\mathrm{ord}_k}{\to} & \frac{1}{c!}\mathbb{Z}_{\geq 0} \\ \| & & & & \cup \\ \mathrm{Sing}(\mathfrak{a}_k^\lambda, b^\lambda) & & \stackrel{\mathrm{ord}_k^\lambda}{\to} & & \frac{1}{b^\lambda}\mathbb{Z}_{\geq 0}. \end{array}$$

(ii) The invariant $w\text{-}\mathrm{ord}_k : F_k \to \frac{1}{c!}\mathbb{Z}\geq 0$ *is a function defined over* F_k *such that for each* λ *it restricts to the invariant* $w\text{-}\mathrm{ord}_k^\lambda$ *of the basic object* $(\widetilde{W_k^\lambda}, (\mathfrak{a}_k^\lambda, b^\lambda), \widetilde{E_k^\lambda})$*, i.e., we have the following commutative diagram*

$$\begin{array}{ccccc} F_k \cap W_k^\lambda & \hookrightarrow & F_k & \stackrel{w\text{-}\mathrm{ord}_k}{\to} & \frac{1}{c!}\mathbb{Z}_{\geq 0} \\ \| & & & & \cup \\ \mathrm{Sing}(\mathfrak{a}_k^\lambda, b^\lambda) & & \stackrel{w\text{-}\mathrm{ord}_k^\lambda}{\to} & & \frac{1}{b^\lambda}\mathbb{Z}_{\geq 0}. \end{array}$$

(iii) First note that in order to define the invariant t_k *we require the following extra condition* $(\heartsuit)$ *on the sequence of transformations and smooth morphisms of general basic objects*

$$(\heartsuit) \quad \left\{ \begin{array}{l} Y_{i-1} \subset \underline{\mathrm{Max}}\ w\text{-}\mathrm{ord}_{i-1}(\subset F_{i-1}) \\ \textit{whenever } \pi_i \textit{ is a transformation with center } Y_{i-1} \end{array} \right\}$$

where

$$\underline{\mathrm{Max}}\ w\text{-}\mathrm{ord}_{i-1} = \{p \in F_{i-1}; w\text{-}\mathrm{ord}_{i-1}(p) = \max w\text{-}\mathrm{ord}_{i-1}\}$$
$$\max w\text{-}\mathrm{ord}_{i-1} = \max\{w\text{-}\mathrm{ord}_{i-1}(p); p \in F_{i-1}\}.$$

Under condition (♡) it follows that we have inequalities (See Proposition 1-12.)

$$\begin{gathered}\max w\text{-}\mathrm{ord}_0 \geq \max w\text{-}\mathrm{ord}_1 \geq \cdots \\ \max w\text{-}\mathrm{ord}_{i-1} \geq \max w\text{-}\mathrm{ord}_i \\ \cdots \geq \max w\text{-}\mathrm{ord}_{k-1} \geq \max w\text{-}\mathrm{ord}_k.\end{gathered}$$

Let k_o be the index so that

$$\max w\text{-}\mathrm{ord}_{k_o-1} > \max w\text{-}\mathrm{ord}_{k_o} = \cdots = \max w\text{-}\mathrm{ord}_k.$$

(We let $k_o = 0$ if $\max w\text{-}\mathrm{ord}_0 = \cdots = \max w\text{-}\mathrm{ord}_k$.) Set $E_k = E_k^- \cup E_k^+$ where $E_k^- = \{H_1, ..., H_r, ..., H_{r+k_o}\}$ as a subset of $E_k = \{H_1, ..., H_r, ..., H_{r+k_o}, ..., H_{r+k}\}$ and where E_k^+ is the complement of E_k^- in E_k. (Look also at the convention explained in Definition 1-8 (iii).)

The invariant $t_k : F_k \to \frac{1}{c!}\mathbb{Z}_{\geq 0} \times \mathbb{Z}_{\geq 0}$ is a function defined over F_k such that

$$t_k(p) = (w\text{-}\mathrm{ord}_k(p), n_k(p)) \text{ for } p \in F_k$$

where

$$n_k(p) = \begin{cases} \#\{H_i \in E_k; p \in H_i\} \text{ if } w\text{-}\mathrm{ord}_k(p) < \max w\text{-}\mathrm{ord}_k \\ \#\{H_i \in E_k^-; p \in H_i\} \text{ if } w\text{-}\mathrm{ord}_k(p) = \max w\text{-}\mathrm{ord}_k. \end{cases}$$

Moreover, for each λ it restricts to the invariant t_k^λ of the basic object $(\widetilde{W_k^\lambda}, (\mathfrak{a}_k^\lambda, b^\lambda), \widetilde{E_k^\lambda})$, i.e., we have the following commutative diagram

$$\begin{array}{ccc} F_k \cap W_k^\lambda & \hookrightarrow \ F_0 \xrightarrow{t_k} & \frac{1}{c!}\mathbb{Z}_{\geq 0} \times \mathbb{Z}_{\geq 0} \\ \| & & \cup \\ \mathrm{Sing}(\mathfrak{a}_k^\lambda, b^\lambda) & \xrightarrow{t_k^\lambda} & \frac{1}{b^\lambda}\mathbb{Z}_{\geq 0} \times \mathbb{Z}_{\geq 0}. \end{array}$$

(iv) Suppose

$$\max\ w\text{-}\mathrm{ord}_k = 0.$$

Then the invariant $\Gamma_k : F_k \to \mathbb{Z}_{\geq -\dim W_k} \times \frac{1}{c!}\mathbb{Z}_{\geq 0} \times \mathbb{Z}_{\geq 0}^{\dim W_k}$ is a function defined over F_k such that for each λ it restricts to the invariant Γ_k^λ of the monomial basic object $(\widetilde{W_k^\lambda}, (\mathfrak{a}_k^\lambda, b^\lambda), \widetilde{E_k^\lambda})$ (shrinking $\widetilde{W_k^\lambda}$ to an open neighborhood of $\mathrm{Sing}(\mathfrak{a}_k^\lambda, b^\lambda) = F_k \cap W_k^\lambda$ if necessary (cf. Corollary 2-7)), i.e., we have the following commutative diagram

$$\begin{array}{ccc} F_k \cap W_k^\lambda & \hookrightarrow \ F_k \xrightarrow{\Gamma_k} & \mathbb{Z}_{\geq -\dim W_k} \times \frac{1}{c!}\mathbb{Z}_{\geq 0} \times \mathbb{Z}_{\geq 0}^{\dim W_k} \\ \| & & \cup \\ \mathrm{Sing}(\mathfrak{a}_k^\lambda, b^\lambda) & \xrightarrow{\Gamma_k^\lambda} & \mathbb{Z}_{\geq -\dim \widetilde{W_k^\lambda}} \times \frac{1}{b^\lambda}\mathbb{Z}_{\geq 0} \times \mathbb{Z}_{\geq 0}^{\dim \widetilde{W_k^\lambda}}. \end{array}$$

Finally, the invariants $\mathrm{ord}_k, w\text{-}\mathrm{ord}_k, t_k$ *and* Γ_k *are determined purely in terms of the collection* $\mathfrak{C}_k$ *of sequences of transformations and smooth morphisms represented by the general basic object* $(\mathcal{F}_k, (W_k, E_k))$*, in terms of the original sequence*

$$(F_0, (W_0, E_0)) \leftarrow \cdots \leftarrow (F_k, (W_k, E_k)),$$

and in terms of the specified dimension $d_k = \dim W_k - \dim W_0 + d_0$ *of the structure of the general basic object* $(\mathcal{F}_k, (W_k, E_k))$*, but free of the presentation using charts.*

Proof.

(i) Since the invariant ord_k depends only on the general basic object $(\mathcal{F}_k, (W_k, E_k))$ and not on the sequence, in order to avoid the complication which may be caused by the subscripts, we prove that the invariant ord_0 exists for the general basic object $(\mathcal{F}_0, (W_0, E_0))$ with the required property.

It suffices to show that the functions $\{\mathrm{ord}_0^\lambda\}$, defined on the individual charts as in Definition 1-10 (i), patch up. That is to say, it suffices to show that for any closed point $x_0 \in F_0$ and commutative diagrams of the form

$$\begin{array}{ccc}
x_0^\lambda \in \mathrm{Sing}(\mathfrak{a}_0^\lambda, b^\lambda) \subset & & \widetilde{W_0^\lambda} \\
 & & \cap \\
\Big\downarrow j_0^\lambda & & W_0^\lambda \\
 & & \cup \\
x_0 & \in & W_0^\lambda \cap W_0^{\lambda'} \\
 & & \cap \\
\Big\uparrow j_0^{\lambda'} & & W_0^{\lambda'} \\
 & & \cup \\
x_0^{\lambda'} \in \mathrm{Sing}(\mathfrak{a}_0^{\lambda'}, b^{\lambda'}) \subset & & \widetilde{W_0^{\lambda'}}
\end{array}$$

we have

$$\mathrm{ord}_0^\lambda(x_0^\lambda) = \frac{\nu_{x_0^\lambda}(\mathfrak{a}_0^\lambda)}{b^\lambda} = \frac{\nu_{x_0^{\lambda'}}(\mathfrak{a}_0^{\lambda'})}{b^{\lambda'}} = \mathrm{ord}_0^{\lambda'}(x_0^{\lambda'}).$$

We will show that the number

$$\frac{\nu_{x_0^\lambda}(\mathfrak{a}_0^\lambda)}{b^\lambda}$$

can be determined purely in terms of the collection $\mathfrak{C}$ of transformations and smooth morphisms representing the general basic object $(\mathcal{F}_0, (W_0, E_0))$ and hence is independent of λ.

The method of the proof below is what Encinas and Villamayor call "**Hironaka's trick**" and this is the only place in the paper where we make use of smooth morphisms (other than open immersions) in the sequences in the collection represented by a general basic object.

We construct a sequence of transformations and smooth morphisms in the following manner. (A warning to the reader: Though we use the same notation, the

sequence we construct below has nothing to do with the original sequence described in the statement of Definition-Proposition 4-5.)

Step 1. First we consider the following smooth morphism, which is nothing but the projection onto the first factor from the product with $\mathbb{A}^1$

$$W_0 \stackrel{\pi_1}{\leftarrow} W_1 = W_0 \times \mathbb{A}^1,$$

where we denote $L_1 = \pi_1^{-1}(x_0)$ and choose a point $x_1 = (x_0, 0) \in L_1 \subset W_0 \times \mathbb{A}^1 = W_1$. For each λ, we have the corresponding smooth morphisms

$$(\widetilde{W_0^\lambda}, (\mathfrak{a}_0^\lambda, b^\lambda), \widetilde{E_0^\lambda}) \stackrel{\pi_1^\lambda}{\leftarrow} (\widetilde{W_1^\lambda}, (\mathfrak{a}_1^\lambda, b^\lambda), \widetilde{E_0^\lambda}) = (\widetilde{W_0^\lambda} \times \mathbb{A}^1, (\mathfrak{a}_0^\lambda \mathcal{O}_{\widetilde{W_1^\lambda}}, b^\lambda), \widetilde{E_0^\lambda} \times \mathbb{A}^1)$$

where, for the indices λ with $x_0 = x_0^\lambda \in F_0 \cap W_0^\lambda \subset \widetilde{W_0^\lambda} \subset W_0^\lambda$, we denote $L_1^\lambda = {\pi_1^\lambda}^{-1}(x_0^\lambda)$ and the corresponding point of choice by $x_1^\lambda = (x_0^\lambda, 0) \in L_1^\lambda \subset \widetilde{W_0^\lambda} \times \mathbb{A}^1 = \widetilde{W_1^\lambda}$.

Step 2. Secondly we consider the sequence of transformations of pairs

$$(W_1, E_1) \stackrel{\pi_2}{\leftarrow} \cdots \stackrel{\pi_N}{\leftarrow} (W_N, E_N)$$

where inductively
π_2 is the blowup at $x_1 \subset L_1 \subset F_1$,
and for $i \geq 3$
π_i is the blowup at $x_{i-1} = L_{i-1} \cap H_{r+i-1} \subset L_{i-1} \subset F_{i-1}$ with H_{r+i-1} being the exceptional divisor for π_{i-1} and L_{i-1} being the strict transform of L_1.

For any λ with $W_1^\lambda \ni x_1$ (i.e., $W_0^\lambda \ni x_0$), we have the corresponding sequence of transformations of basic objects in the charts

$$(\widetilde{W_1^\lambda}, (\mathfrak{a}_1^\lambda, b^\lambda), \widetilde{E_1^\lambda}) \stackrel{\pi_2^\lambda}{\leftarrow} \cdots \stackrel{\pi_N^\lambda}{\leftarrow} (\widetilde{W_N^\lambda}, (\mathfrak{a}_N^\lambda, b^\lambda), \widetilde{E_N^\lambda})$$

where
π_2^λ is the blowup at $x_1^\lambda \subset L_1^\lambda \subset \mathrm{Sing}(\mathfrak{a}_1^\lambda, b^\lambda)$,
and for $i \geq 3$
π_i^λ is the blowup at $x_{i-1}^\lambda = L_{i-1}^\lambda \cap \widetilde{H_{r+i-1}^\lambda} \subset L_{i-1}^\lambda \subset \mathrm{Sing}(\mathfrak{a}_{i-1}^\lambda, b^\lambda)$ with $\widetilde{H_{r+i-1}^\lambda}$ being the exceptional divisor for π_{i-1}^λ and L_{i-1}^λ being the strict transform of L_1^λ.

Note that under the inclusion $\widetilde{W_{i-1}^\lambda} \subset W_{i-1}^\lambda$ we identify

$$x_{i-1}^\lambda = x_{i-1}, L_{i-1}^\lambda = L_{i-1}, \widetilde{H_{r+i-1}^\lambda} = H_{r+i-1} \cap \widetilde{W_{i-1}^\lambda}.$$

By conditions (GB-0) and (GB-2) this gives rise to a sequence in the collection $\mathfrak{C}$ of the general basic object $(\mathcal{F}_0, (W_0, E_0))$

$$(F_0, (W_0, E_0)) \stackrel{\pi_1}{\leftarrow} (F_1, (W_1, E_1)) \stackrel{\pi_2}{\leftarrow} \cdots \stackrel{\pi_N}{\leftarrow} (F_N, (W_N, E_N))$$

where

$$F_i = \cup \mathrm{Sing}(\mathfrak{a}_i^\lambda, b^\lambda)$$

and for each λ we have

$$F_i \cap W_i^\lambda = F_i \cap \widetilde{W_i^\lambda} = \mathrm{Sing}(\mathfrak{a}_i^\lambda, b^\lambda).$$

We compute the transformations of ideals

$$\mathfrak{a}_2^\lambda = I(\widetilde{H_{r+2}^\lambda})^{(\beta^\lambda - b^\lambda)}\overline{\mathfrak{a}_2^\lambda} \text{ where } \beta^\lambda = \nu_{x_0^\lambda}(\mathfrak{a}_0^\lambda),$$

since

$$\nu_{\xi_1}(\overline{\mathfrak{a}_1^\lambda}) = \nu_{\xi_1}(\mathfrak{a}_1^\lambda) = \beta^\lambda \quad \forall \xi_1 \in L_1^\lambda.$$

Note that

$$\nu_{\xi_2}(\overline{\mathfrak{a}_2^\lambda}) = \beta^\lambda \quad \forall \xi_2 \in L_2^\lambda.$$

In fact, it is clear that

$$\nu_{\xi_2}(\overline{\mathfrak{a}_2^\lambda}) = \beta^\lambda \quad \forall \xi_2 \in L_2^\lambda \setminus x_2^\lambda$$

and hence by the upper semi-continuity

$$\nu_{\xi_2}(\overline{\mathfrak{a}_2^\lambda}) \geq \beta^\lambda \text{ for } \xi_2 = x_2^\lambda.$$

On the other hand, applying Proposition 1-12 (ii) locally, we conclude

$$\nu_{x_2^\lambda}(\overline{\mathfrak{a}_2^\lambda}) \leq \nu_{x_1^\lambda}(\overline{\mathfrak{a}_1^\lambda}) = \nu_{x_1^\lambda}(\mathfrak{a}_1^\lambda) = \beta^\lambda.$$

Therefore, inductively locally around x_i^λ for $i = 2, ..., N$, we compute

$$\mathfrak{a}_i^\lambda = I(\widetilde{H_{r+i}^\lambda})^{(i-1)(\beta^\lambda - b^\lambda)}\overline{\mathfrak{a}_i^\lambda}$$

with

$$\nu_{\xi_i}(\overline{\mathfrak{a}_i^\lambda}) = \beta^\lambda \quad \forall \xi_i \in L_i^\lambda.$$

Therefore, we conclude that

$$\begin{aligned} & \dim F_N \cap H_{r+N} = d = (d+1) - 1 \\ \iff \quad & \widetilde{H_{r+N}^\lambda} = H_{r+N} \cap \widetilde{W_N^\lambda} \subset F_N^\lambda \\ \iff \quad & (N-1)(\beta^\lambda - b^\lambda) \geq b^\lambda. \end{aligned}$$

Remark that the condition on the first line is determined purely in terms of the collection $\mathfrak{C}$ represented by the general basic object $(\mathcal{F}_0, (W_0, E_0))$ and the condition on the third line is a numerical one about the number b^λ.

Step 3. Therefore we conclude that

$$\begin{aligned} & \beta^\lambda = b^\lambda \\ \iff \quad & \widetilde{H_{r+N}^\lambda} = H_{r+N} \cap \widetilde{W_N^\lambda} \not\subset F_N^\lambda \text{ for all } N \in \mathbb{N} \\ \iff \quad & \dim F_N \cap H_{r+N} < d = (d+1) - 1 \text{ for all } N \in \mathbb{N} \end{aligned}$$

and that

$$\begin{aligned} & \beta^\lambda > b^\lambda \\ \Longleftrightarrow\quad & \widetilde{H^\lambda_{r+N}} = H_{r+N} \cap \widetilde{W^\lambda_N} \subset F^\lambda_N \text{ for all sufficiently large } N \in \mathbb{N} \\ \Longleftrightarrow\quad & \dim F_N \cap H_{r+N} = d = (d+1) - 1 \text{ for all sufficiently large } N \in \mathbb{N}. \end{aligned}$$

In the latter case, we consider the further extension of the sequence of transformations of pairs

$$(W_N, E_N) \overset{\pi_{N+1}}{\longleftarrow} \cdots \overset{\pi_{N+S}}{\longleftarrow} (W_{N+S}, E_{N+S})$$

where π_{N+i} is the blowup with center $Y_{N+i-1} = F_{N+i-1} \cap H_{r+N+i-1}$.

For each λ there corresponds the sequence of transformations of basic objects

$$(\widetilde{W^\lambda_N}, (\mathfrak{a}^\lambda_N, b^\lambda), \widetilde{E^\lambda_N}) \overset{\pi^\lambda_{N+1}}{\longleftarrow} \cdots \overset{\pi_{N+S}}{\longleftarrow} (\widetilde{W^\lambda_{N+S}}, (\mathfrak{a}^\lambda_{N+S}, b^\lambda), \widetilde{E^\lambda_{N+S}})$$

where π^λ_{N+i} is the blowup with center

$$\begin{aligned} \widetilde{H^\lambda_{r+N+i-1}} &= Y_{N+i-1} \cap \widetilde{W^\lambda_{N+i-1}} \\ &= F_{N+i-1} \cap H_{r+N+i-1} \cap \widetilde{W^\lambda_{N+i-1}} \\ &= F_{N+i-1} \cap H_{r+N+i-1} \cap W^\lambda_{N+i-1}. \end{aligned}$$

Note that these transformations are set-theoretically nothing but identities with

$$\widetilde{H^\lambda_{r+N}} = \widetilde{H^\lambda_{r+N+1}} = \cdots = \widetilde{H^\lambda_{r+N+S-1}}.$$

Therefore, by condition (GB-2) this gives rise to a sequence of transformations in the collection $\mathfrak{C}$ of the general basic object $(\mathcal{F}_0, (W_0, E_0))$ as long as $\widetilde{H^\lambda_{r+N+i-1}} \subset \mathrm{Sing}(\mathfrak{a}^\lambda_{N+i-1}, b^\lambda)$ for $i = 1, ..., S$, the condition which translates into the following equivalent conditions:

$$\begin{aligned} & (N-1)(\beta^\lambda - b^\lambda) - (i-1)b^\lambda \geq b^\lambda \text{ for } i = 1, ..., S \\ \Longleftrightarrow\quad & (N-1)(\beta^\lambda - b^\lambda) - (S-1)b^\lambda \geq b^\lambda \\ \Longleftrightarrow\quad & (N-1)(\beta^\lambda - b^\lambda) \geq S b^\lambda \\ \Longleftrightarrow\quad & \left[\frac{(N-1)(\beta^\lambda - b^\lambda)}{b^\lambda}\right] \geq S, \end{aligned}$$

where $[\ \ ]$ is the Gauss symbol, representing the integer $[x] = \alpha \in \mathbb{N}$ such that $\alpha \leq x < \alpha + 1$.

Therefore, we finally conclude by conditions (GB-0) (GB-1) and (GB-2) that for a fixed sufficiently large $N \in \mathbb{N}$,

$$\left[\frac{(N-1)(\beta^\lambda - b^\lambda)}{b^\lambda}\right]$$

is characterized as the largest integer S_N such that the sequence described as above of one smooth morphism followed by the transformations

$$\begin{aligned}(F_0,(W_0,E_0)) \leftarrow (F_1,(W_1,E_1)) \leftarrow \cdots \leftarrow (F_N,(W_N,E_N))\\ \leftarrow (F_{N+1},(W_{N+1},E_{N+1})) \leftarrow \cdots \leftarrow (F_{N+S_N},(W_{N+S_N},E_{N+S_N}))\end{aligned}$$

is in the collection $\mathfrak{C}$ represented by the general basic object $(\mathcal{F}_0,(W_0,E_0))$, and hence that the number

$$\frac{\nu_{x_0^\lambda}(\mathfrak{a}_0^\lambda)}{b^\lambda} - 1 = \frac{\beta^\lambda}{b^\lambda} - 1 = \lim_{N\to\infty} \frac{1}{N-1}\left[\frac{(N-1)(\beta^\lambda - b^\lambda)}{b^\lambda}\right] = \lim_{N\to\infty}\frac{1}{N-1}S_N$$

is characterized purely in terms of the collection $\mathfrak{C}$ represented by the general basic object $(\mathcal{F}_0,(W_0,E_0))$ and hence of independent of λ.

This completes the proof of (i), verifying that ord_0 is a well-defined function on the singular locus F_0 of the general basic object $(\mathcal{F}_0,(W_0,E_0))$. (Therefore, bringing back the subscripts right, we complete the proof that ord_k is a well-defined function on the singular locus F_k of the general basic object $(\mathcal{F}_k,(W_k,E_k))$.

(ii) As in the proof of (i), it suffices to show that the functions $\{w\text{-}\mathrm{ord}_k^\lambda\}$, defined on the individual charts as in Definition 1-10 (ii), patch up. That is to say, it suffices to show that for any closed point $x_k \in F_k$ and commutative diagram of the form

$$\begin{array}{ccc}
x_k^\lambda \in \mathrm{Sing}(\mathfrak{a}_k^\lambda, b^\lambda) \subset & & \widetilde{W_k^\lambda}\\
 & & \cap\\
\Big\downarrow j_k^\lambda & & W_k^\lambda\\
 & & \cup\\
x_k & \in & W_k^\lambda \cap W_k^{\lambda'}\\
 & & \cap\\
\Big\uparrow j_k^{\lambda'} & & W_k^{\lambda'}\\
 & & \cup\\
x_k^{\lambda'} \in \mathrm{Sing}(\mathfrak{a}_k^{\lambda'}, b^{\lambda'}) \subset & & \widetilde{W_k^{\lambda'}}
\end{array}$$

we have

$$w\text{-}\mathrm{ord}_k^\lambda(x_k^\lambda) = \frac{\nu_{x_k^\lambda}(\overline{\mathfrak{a}_k^\lambda})}{b^\lambda} = \frac{\nu_{x_k^{\lambda'}}(\overline{\mathfrak{a}_k^{\lambda'}})}{b^{\lambda'}} = w\text{-}\mathrm{ord}_k^{\lambda'}(x_k^{\lambda'}).$$

We will show that the number

$$w\text{-}\mathrm{ord}_k^\lambda(x_k^\lambda)$$

can be determined purely in terms of the invariants ord_i for the general basic objects $(\mathcal{F}_i,(W_i,E_i))$ for $i = 0,...,k$ and in terms of the sequence

$$(F_0,(W_0,E_0)) \leftarrow \cdots \leftarrow (F_k,(W_k,E_k)),$$

and hence independent of λ.

Claim 4-6. *We have the formula*

$(*)$ $w\text{-ord}_k^\lambda(x_k) = \text{ord}_k(x_k) - \Sigma_{j=1}^k \Sigma_{H_{r+j,l} \subset H_{r+j}} \{\text{ord}_{i_{H_{r+j,l}}}(\eta_{Y_{H_{r+j,l}}}) - 1\} \cdot \epsilon_{H_{r+j,l}, x_k}$

where

$E_k = \{H_1, ..., H_r, H_{r+1}, ..., H_{r+k}\}$ (cf. the convention in Definition 1-8 (iii)),

$H_{r+j,l}$ are the irreducible components of H_{r+j} with the generic points $\eta_{H_{r+j,l}}$,

the number $i_{H_{r+j,l}}$ is the maximum of such i that $\overline{\pi_{i+1} \circ \cdots \circ \pi_k(\eta_{H_{r+j,l}})}$ is an irreducible component of the center Y_i (See the convention explained in Definition 1-8 (iii).),

$Y_{H_{r+j,l}} = \pi_{i_{H_{r+j,l}}+1} \circ \cdots \circ \pi_k(H_{r+j,l})$ with the generic point $\eta_{Y_{H_{r+j,l}}}$, and

$$\epsilon_{H_{r+j,l}, x_k} = \begin{cases} 0 \text{ if } x_k \notin H_{r+j,l} \\ 1 \text{ if } x_k \in H_{r+j,l}. \end{cases}$$

In particular, since the right hand side does not depend on λ, $w\text{-ord}_k^\lambda(x_k)$ is also independent of λ as desired.

Proof.

Observe

$$\mathfrak{a}_k^\lambda = I(\widetilde{H_{r+1}^\lambda})^{a_{r+1}^\lambda} \cdots I(\widetilde{H_{r+k}^\lambda})^{a_{r+k}} \overline{\mathfrak{a}_k^\lambda}$$

where $\widetilde{H_{r+j}^\lambda} = H_{r+j} \cap \widetilde{W_k^\lambda}$ and a_{r+j}^λ are multi-indices so that

$$I(\widetilde{H_{r+j}^\lambda})^{a_{r+j}^\lambda} = \prod I(\widetilde{H_{r+j,l}^\lambda})^{a_{r+j,l}^\lambda}$$

where $\widetilde{H_{r+j,l}^\lambda} = H_{r+j,l} \cap \widetilde{W_j^\lambda}$ are the irreducible components of $\widetilde{H_{r+j}^\lambda}$. (Note that the irreducible components $H_{r+j,l}$ of H_{r+j} are in one-to-one correspondence with the irreducible components $\widetilde{H_{r+j,l}}$ of $\widetilde{H_{r+j}}$.)

Therefore, we conclude

$(*)_\lambda$ $\text{ord}_k^\lambda(x_k^\lambda) = w\text{-ord}_k(x_k^\lambda) + \Sigma_{j=1}^k \Sigma_{\widetilde{H_{r+j,l}} \subset \widetilde{H_{r+j}}} \{\text{ord}_{i_{\widetilde{H_{r+j,l}}}}(\eta_{Y_{\widetilde{H_{r+j,l}}}}) - 1\} \cdot \epsilon_{\widetilde{H_{r+j,l}}, x_k^\lambda}$

where

$\widetilde{H_{r+j,l}}$ are the irreducible components of $\widetilde{H_{r+j}}$ with the generic points $\eta_{\widetilde{H_{r+j,l}}}$,

the number $i_{\widetilde{H_{r+j,l}}}$ is the maximum of such i that $\overline{\pi_{i+1}^\lambda \circ \cdots \pi_k^\lambda(\eta_{\widetilde{H_{r+j,l}}})}$ is an irreducible component of Y_i,

$Y_{\widetilde{H_{r+j,l}}} = \pi^\lambda_{i_{\widetilde{H_{r+j,l}}}+1} \circ \cdots \circ \pi_k^\lambda(\widetilde{H_{r+j,l}})$ with the generic point $\eta_{Y_{\widetilde{H_{r+j,l}}}}$, and

$$\epsilon_{\widetilde{H_{r+j,l}}, x_k^\lambda} = \begin{cases} 0 \text{ if } x_k^\lambda \notin \widetilde{H_{r+j,l}} \\ 1 \text{ if } x_k^\lambda \in \widetilde{H_{r+j,l}}. \end{cases}$$

Note that, denoting $\pi^{-1}_{i_{\widetilde{H_{r+j,l}}}+1}(Y_{\widetilde{H_{r+j,l}}})$ by $\widetilde{H_{r+j,l}}$ and its generic point $\eta_{\widetilde{H_{r+j,l}}}$ by abuse of notation, we compute

$$\begin{aligned}\frac{a^\lambda_{r+j,l}}{b^\lambda} = \frac{\nu_{\eta_{\widetilde{H_{r+j,l}}}}(\mathfrak{a}^\lambda_k)}{b^\lambda} &= \frac{\nu_{\eta_{\widetilde{H_{r+j,l}}}}(\mathfrak{a}^\lambda_{i_{\widetilde{H_{r+j,l}}}+1})}{b^\lambda} \\ &= \frac{\nu_{\eta_{Y_{\widetilde{H_{r+j,l}}}}}(\mathfrak{a}^\lambda_{i_{\widetilde{H_{r+j,l}}}}) - b^\lambda}{b^\lambda} \\ &= \mathrm{ord}_{i_{H_{r+j,l}}}(\eta_{Y_{\widetilde{H_{r+j,l}}}}) - 1.\end{aligned}$$

Remark that for the corresponding irreducible components $\widetilde{H_{r+j,l}} \subset H_{r+j,l}$, the numbers and the generic points of the center coincide

$$\begin{aligned} i_{\widetilde{H_{r+j,l}}} &= i_{H_{r+j,l}} \\ \eta_{Y_{\widetilde{H_{r+j,l}}}} &= \eta_{Y_{H_{r+j,l}}}. \end{aligned}$$

The formula $(*)$ in the claim now follows from this remark and the formula $(*)_\lambda$.

(iii) Since w-ord is a well-defined invariant on a general basic object by (ii), and since the inequalities

$$\begin{aligned} \max w\text{-ord}_0 &\geq \max w\text{-ord}_1 \geq \cdots \\ \max w\text{-ord}_{i-1} &\geq \max w\text{-ord}_i \\ \cdots \geq \max w\text{-ord}_{k-1} &\geq \max w\text{-ord}_k. \end{aligned}$$

follow from those for the basic objects under condition $(\heartsuit)$, the invariant $t_k : F_k \to \frac{1}{c!}\mathbb{Z}_{\geq 0} \times \mathbb{Z}_{\geq 0}$ is a function well-defined globally on F_k.

In order to verify that the restriction $t_k|_{F_k \cap W^\lambda_k}$ coincides with t^λ_k defined on $\mathrm{Sing}(\mathfrak{a}^\lambda_k, b^\lambda) = F_k \cap W^\lambda_k$, one has only to observe that t_k has the local description identical to the one given in Remark 1-11 (v), which is easily seen to coincide with the local description of t^λ_k also given in Remark 1-11 (v).

(iv) Let $E_k = \{H_1, ..., H_r, H_{r+1}, ..., H_{r+k}\}$. To a point $p \in H_{r+j}$ we assign the following number $\alpha_{r+j}(p)$

$$\alpha_{r+j}(p) = \mathrm{ord}_{i_{H_{r+j,l}}}(\eta_{Y_{\widetilde{H_{r+j,l}}}}) - 1,$$

where $H_{r+j,l}$ is the irreducible component of H_{r+j} containing $p \in H_{r+j,l}$ (cf. the formula for $\frac{a^\lambda_{r+j,l}}{b^\lambda}$ at the end of the proof for (ii)).

We define the invariant

$$\Gamma_k : F_k \to \mathbb{Z}_{\geq -d_k} \times \frac{1}{c!}\mathbb{Z}_{\geq 0} \times \mathbb{Z}_{\geq 0}{}^{d_k},$$

where $d_k = \dim \widetilde{W^\lambda_k}$ (independent of λ), to be a function defined over F_k such that

$$\Gamma_k(p) = (\Gamma_{k1}(p), \Gamma_{k2}(p), \Gamma_{k3}(p)) \text{ for } p \in F_k$$

where

$$
\begin{aligned}
-\Gamma_{k1}(p) &= \min\{n; \exists r+j_1, ..., r+j_n \text{ s.t. } \alpha_{r+j_1}(p) + \cdots + \alpha_{r+j_n}(p) \geq 1, \\
&\qquad p \in H_{r+j_1} \cap \cdots \cap H_{r+j_n}\} \\
\Gamma_{k2}(p) &= \max\{\alpha_{r+j_1}(p) + \cdots + \alpha_{r+j_n}(p); n = -\Gamma_{k1}(p), \alpha_{r+j_1}(p) + \cdots + \alpha_{r+j_n}(p) \geq 1, \\
&\qquad p \in H_{r+j_1} \cap \cdots \cap H_{r+j_n}\} \\
\Gamma_{k3}(p) &= \max\{(r+j_1, ..., r+j_n); n = -\Gamma_{k1}(p), \Gamma_{k2}(p) = \alpha_{r+j_1}(p) + \cdots + \alpha_{r+j_n}(p), \\
&\qquad p \in H_{r+j_1} \cap \cdots \cap H_{r+j_n}, r+j_1 \geq \cdots \geq r+j_n\}
\end{aligned}
$$

with the maximum taken with respect to the lexicographical order.

We identify $(r+j_1, ..., r+j_n)$ with $(r+j_1, ..., r+j_n, 0, ..., 0) \in \mathbb{Z}_{\geq 0}{}^{d_k}$.

(We order the values of Γ_k according to the lexicographical order given to $\mathbb{Z}_{\geq -d_k} \times \frac{1}{c!}\mathbb{Z}_{\geq 0} \times \mathbb{Z}_{\geq 0}{}^{d_k}$.)

It is clear from the above description that for each λ the invariant Γ_k restricts to the invariant Γ_k^λ of the monomial basic object $(\widetilde{W_k^\lambda}, (\mathfrak{a}_k^\lambda, b^\lambda), \widetilde{E_k^\lambda})$ (cf. Definition 2-3) (shrinking $\widetilde{W_k^\lambda}$ to an open neighborhood of $\mathrm{Sing}(\mathfrak{a}_k^\lambda, b^\lambda) = F_k \cap W_k^\lambda$ if necessary (cf. Corollary 2-7)).

Finally, from the proof of (i), it is clear that the invariant ord_k is purely determined in terms of the collection $\mathfrak{C}_k$ of sequences of transformations and smooth morphisms represented by the general basic object $(\mathcal{F}_k, (W_k, E_k))$, once the dimension $d_k = \dim W_k - \dim W_0 + d_0$ of the structure is specified. From Claim 4-6, it follows that the invariant $w\text{-}\mathrm{ord}_k$ is purely determined in terms of $\mathrm{ord}_0, ..., \mathrm{ord}_k$ and by looking at the set-theoretical behavior of $E_1, ..., E_k$ in the original sequence. The invariant t_k is purely detrmined in terms of $w\text{-}\mathrm{ord}_0, ..., w\text{-}\mathrm{ord}_k$ and by looking at the set-theoretical behavior of $E_1, ..., E_k$ in the original sequence. The invariant Γ_k is purely determined in terms of ord_k and by looking at the set-theoretical behavior of $E_1, ..., E_k$ in the original sequence. This verifies the "Finally" part of definition-Proposition 4-5.

This completes the proof of Definition-Proposition 4-5.

Remark 4-7.

A general basic object $(\mathcal{F}, (W, E))$ sometimes can have a d-dimensional structure as well as a d'-dimensional structure for two different numbers $d \neq d'$. That is to say, we can have two different sets of charts $\{(\widetilde{W^\lambda}, (\mathfrak{a}^\lambda, b^\lambda), \widetilde{E^\lambda})\}_{\lambda \in \Lambda}$ and $\{(\widetilde{W^\mu}, (\mathfrak{b}^\mu, c^\mu), \widetilde{E^\mu})\}_{\mu \in M}$ being of different dimensions d and d', i.e., $\dim \widetilde{W^\lambda} = d \neq d' = \widetilde{W^\mu}$, but giving rise to the same collection $\mathfrak{C}$ of sequences of smooth morphisms and transformations of pairs with specified closed subsets, represented by the general basic object $(\mathcal{F}, (W, E))$.

The invariants ord and w-ord DO depend on the specification of the dimension of the structure of your choice, as the following example demonstrates (cf. Remark 3-7 (ii)).

Take a basic object

$$(W, (J, b), E) = (\mathbb{A}^2, (\langle x^2 - y^3 \rangle, 2), \emptyset),$$

which defines a general basic object with a 2-dimensional structure via Remark 4-2 (ii).

By the key inductive lemma, the same general basic object has a $2-1=1$-dimensional structure with a (global) chart

$$(\mathbb{A}^1 = \{x=0\} = \operatorname{Spec} k[y], (\langle (y^3)^{\frac{2!}{2-0}}, (y^2)^{\frac{2!}{2-1}} \rangle = \langle y^3 \rangle, 2!), \emptyset).$$

Denoting by $\mathrm{ord}_0^{(2)}, w\text{-}\mathrm{ord}_0^{(2)}$ the ord$-$ and w-ord$-$invarinats with respect to the 2-dimensional structure of the general basic object (considered to form a trivial sequence by itself), we have

$$\mathrm{ord}_0^{(2)}(0) = w\text{-}\mathrm{ord}_0^{(2)}(0) = \frac{2}{2} = 1.$$

On the other hand, denoting by $\mathrm{ord}_0^{(1)}, w\text{-}\mathrm{ord}_0^{(1)}$ the ord$-$ and w-ord$-$invarinats with respect to the 1-dimensional structure of the general basic object (considered to form a trivial sequence by itself), we have

$$\mathrm{ord}_0^{(1)}(0) = w\text{-}\mathrm{ord}_0^{(1)}(0) = \frac{3}{2!} = \frac{3}{2}.$$

Therefore, theoretically and strictly speaking, it might be more appropriate to put the superscript d_k, such as $\mathrm{ord}_k^{(d_k)}$ or $w\text{-}\mathrm{ord}_k^{(d_k)}$ to indicate the dependence of the invariants ord_k or $w\text{-}\mathrm{ord}_k$ upon the specified dimension d_k of the structure. However, we will omit the superscript for notational simplicity, since little confusion is likely to occur.

Proposition 4-8 (Properties of key invariants for general basic objects). *Let*

$$\begin{aligned} (\mathcal{F}_0, (W_0, E_0)) &\stackrel{\pi_1}{\leftarrow} (\mathcal{F}_1, (W_1, E_1)) \stackrel{\pi_2}{\leftarrow} \cdots \\ (\mathcal{F}_{i-1}, (W_{i-1}, E_{i-1})) &\stackrel{\pi_i}{\leftarrow} (\mathcal{F}_i, (W_i, E_i)) \\ \cdots \stackrel{\pi_{k-1}}{\leftarrow} (\mathcal{F}_{k-1}, (W_{k-1}, E_{k-1}) &\stackrel{\pi_k}{\leftarrow} (\mathcal{F}_k, (W_k, E_k)) \end{aligned}$$

be a sequence of transformations and smooth morphisms of general basic objects.

(i) The invariants ord_k *and* $w\text{-}\mathrm{ord}_k$ *are upper semi-continuous functions.*
(ii) Note first that

$$F_{i-1} \supset \pi_i(F_i) \text{ for } i = 1, ..., k.$$

Suppose that the sequence satisfies condition ($\heartsuit$) *(See Definition-Proposition 4-5 (iii).).*

Then for $i = 1, ..., k$ *we have inequalities*

$$w\text{-}\mathrm{ord}_{i-1}(\xi_{i-1}) \geq w\text{-}\mathrm{ord}_i(\xi_i)$$

where $\xi_i \in F_i$ *and* $\xi_{i-1} = \pi_i(\xi_i) \in F_{i-1}$*, which imply*

$$\max\ w\text{-}\mathrm{ord}_{i-1} \geq \max\ w\text{-}\mathrm{ord}_i.$$

That is to say, we have

$$\max\ w\text{-ord}_0 \geq \max\ w\text{-ord}_1 \geq \cdots$$
$$\max\ w\text{-ord}_{i-1} \geq \max\ w\text{-ord}_i$$
$$\cdots \geq \max\ w\text{-ord}_{k-1} \geq \max\ w\text{-ord}_k.$$

The invariant t_k is an upper semi-continuous function and we have inequalities

$$t_{i-1}(\xi_{i-1}) \geq t_i(\xi_i)$$

where $\xi_i \in F_i$ and $\xi_{i-1} = \pi_i(\xi_i) \in F_{i-1}$, which imply

$$\max\ t_{i-1} \geq \max\ t_i.$$

That is to say, we have

$$\max\ t_0 \geq \max\ t_1 \geq \cdots$$
$$\max\ t_{i-1} \geq \max\ t_i$$
$$\cdots \geq \max\ t_{k-1} \geq \max\ t_k.$$

Proof.

The proof is identical to the one for basic objects (cf. Proposition 1-12) and left to the reader as an exercise.

Corollary 4-9 (Resolution of singularities of a general basic object with $\max\ w\text{-ord} = 0$**).** *Let*

$$(\mathcal{F}_0, (W_0, E_0)) \leftarrow \cdots \leftarrow (\mathcal{F}_k, (W_k, E_k))$$

be a sequence of transformations and smooth morphisms of general basic objecs.

Suppose $\max\ w\text{-ord}_k = 0$.

Then there exists a sequence of transformations only of general basic objects

$$(\mathcal{F}_k, (W_k, E_k)) \overset{\pi_{k+1}}{\leftarrow} (\mathcal{F}_{k+1}, (W_{k+1}, E_{k+1})) \overset{\pi_{k+2}}{\leftarrow} \cdots$$
$$(\mathcal{F}_{k+i-1}, (W_{k+i-1}, E_{k+i-1})) \overset{\pi_{k+i}}{\leftarrow} (\mathcal{F}_{k+i}, (W_{k+i}, E_{k+i}))$$
$$\cdots \overset{\pi_{k+N-1}}{\leftarrow} (\mathcal{F}_{k+N-1}, (W_{k+N-1}, E_{k+N-1})) \overset{\pi_{k+N}}{\leftarrow} (\mathcal{F}_{k+N}, (W_{k+N}, E_{k+N}))$$

which represents resolution of singularities, i.e.,

$$F_{k+N} = \emptyset,$$

where π_{k+i} for $i = 1, ..., N$ are the transformations with centers

$$Y_{k+i-1} = \underline{\text{Max}}\ \Gamma_{k+i-1} \subset F_{k+i-1}.$$

Proof.

We note that under the specified transformations $\max\ w$-ord remains zero, i.e.,

$$\max\ w\text{-ord}_{k+i} = 0 \text{ for } i = 0, ..., N-1$$

and hence that the invariant Γ_{k+i} is well-defined. The rest of the proof is identical to the one for resolution of singularities for basic objects with $\max\ w\text{-ord} = 0$ and left to the reader as an exercise (cf. Corollary 2-7).

CHAPTER 5. INDUCTIVE ALGORITHM FOR RESOLUTION OF SINGULARITIES OF GENERAL BASIC OBJECTS

This chapter is the culmination of the ideas of Encinas and Villamayor, seeing how we overcome the shortcomings (cf. Remark 3-2) of the key inductive lemma (Lemma 3-1) via the use of the t-invariant and transform the lemma into a genuine inductive algorithm of resolution of singularities of general basic objects.

Theorem 5-1 (Inductive algorithm for resolution of singularities of general basic objects). *Let $(\mathcal{F}_0,(W_0,E_0))$ be a general basic object over $(F_0,(W_0,E_0))$ with a d-dimensional structure given by an open covering $\{W^\lambda\}_{\lambda\in\Lambda}$ and charts $\{(\widetilde{W_0^\lambda},(\mathfrak{a}_0^\lambda,b^\lambda),\widetilde{E_0^\lambda})\}$ of d-dimensional basic objects. Let $\mathfrak{C}$ denote the collection of sequences of transformations and smooth morphisms represented by $(\mathcal{F}_0,(W_0,E_0))$ (cf. Definition 4-1).*

Then there exists an inductive algorithm which provides a seqeuence of trasformations of pairs with specified closed subsets in the collection $\mathfrak{C}$, representing resolution of singularities of the general basic object $(\mathcal{F}_0,(W_0,E_0))$

$$(F_0,(W_0,E_0)) \leftarrow \cdots \leftarrow (F_l,(W_l,E_l)) \text{ with } F_l=\emptyset,$$

by uniquely specifying the centers satisfying the following condition $(\heartsuit')$ for $i=1,...,l$

$$(\heartsuit')\ Y_{i-1}\subset \underline{\mathrm{Max}}\ t_{i-1}\subset \underline{\mathrm{Max}}\ w\text{-}\mathrm{ord}_{i-1}\subset F_{i-1} \text{ if } \max\ w\text{-}\mathrm{ord}_{i-1}>0.$$

The process of the inductive algorithm is described below:

Suppose we have constructed the sequence of transformations up to the k-th stage via the inductive algorithm

$$(F_0,(W_0,E_0)) \leftarrow \cdots \leftarrow (F_k,(W_k,E_k))$$

with the centers satisfying condition $(\heartsuit')$ for $i=1,...,k$

$$(\heartsuit')\ Y_{i-1}\subset \underline{\mathrm{Max}}\ t_{i-1}\subset \underline{\mathrm{Max}}\ w\text{-}\mathrm{ord}_{i-1}\subset F_{i-1} \text{ if } \max\ w\text{-}\mathrm{ord}_{i-1}>0.$$

(Recall that condition $(\heartsuit)$

$$(\heartsuit)\ Y_{i-1}\subset \underline{\mathrm{Max}}\ w\text{-}\mathrm{ord}_{i-1}\subset F_{i-1} \text{ for } i=1,...,k,$$

which obviously follows from condition $(\heartsuit')$, implies inequalities

$$\max\ w\text{-}\mathrm{ord}_0 \geq \cdots \geq \max\ w\text{-}\mathrm{ord}_{i-1} \geq \max\ w\text{-}\mathrm{ord}_i \geq \cdots \geq \max\ w\text{-}\mathrm{ord}_k$$
$$\max\ t_0 \geq \cdots \geq \max\ t_{i-1} \geq \max\ t_i \geq \cdots \geq \max\ t_k.$$

Recall also that the sequence induces an sequence of general basic objects

$$(\mathcal{F}_0,(W_0,E_0)) \leftarrow \cdots \leftarrow (\mathcal{F}_k,(W_k,E_k))$$

as explained in Note 4-3.)

Then we have the following three possibilities:

<u>P1</u> : $F_k = \emptyset$.

Under this possibility, the sequence represents resolution of singularities of the general basic object $(\mathcal{F}_0, (W_0, E_0))$.

<u>P2</u> : $F_k \neq \emptyset$ *and* $\max\ w\text{-ord}_k = 0$.

Under this possibility, we can apply Corollary 4-9 to $(\mathcal{F}_k, (W_k, E_k))$ *and create a sequence of transformations representing resolution of singularities of* $(\mathcal{F}_k, (W_k, E_k))$. *Attaching this to the original sequence, we obtain a sequence representing resolution of singularities of* $(\mathcal{F}_0, (W_0, E_0))$.

<u>P3</u> : $F_k \neq \emptyset$ *and* $\max\ w\text{-ord}_k > 0$.

Under this possibility **<u>P3</u>**, *there are two cases* **Case A** *and* **Case B**.

We denote by $R(1)(\underline{\mathrm{Max}}\ t_k)$ *the* $(d-1)$*-dimensional part (i.e., codimension one with respect to the d-dimensional* $\widetilde{W_k^\lambda}$*'s) of the locus* $\underline{\mathrm{Max}}\ t_k \subset F_k$.

Case A*:* $R(1)(\underline{\mathrm{Max}}\ t_k) \neq \emptyset$.

In this case, $R(1)(\underline{\mathrm{Max}}\ t_k)(\subset \underline{\mathrm{Max}}\ t_k \subset \underline{\mathrm{Max}}\ w\text{-ord}_k \subset F_k)$ *is smooth, open and closed in* $\underline{\mathrm{Max}}\ w\text{-ord}_k$ *(i.e., a union of smooth connected components of* $\underline{\mathrm{Max}}\ w\text{-ord}_k$ *disjoint from each other).*

The locus $R(1)(\underline{\mathrm{Max}}\ t_k) \cap W_k^\lambda$ *is permissible for each* $(\widetilde{W_k^\lambda}, (\mathfrak{a}_k^\lambda, b^\lambda), \widetilde{E_k^\lambda})$, *i.e.,* $R(1)(\underline{\mathrm{Max}}\ t_k) \cap W_k^\lambda \subset \widetilde{W_k^\lambda}$, $R(1)(\underline{\mathrm{Max}}\ t_k) \cap W_k^\lambda$ *is permissible with respect to* $\widetilde{E_k^\lambda}$, *and* $R(1)(\underline{\mathrm{Max}}\ t_k) \cap W_k^\lambda \subset \mathrm{Sing}(\mathfrak{a}_k^\lambda, b^\lambda)$.

Take the transformation of pairs

$$(W_k, E_k) \overset{\pi_{k+1}}{\longleftarrow} (W_{k+1}, E_{k+1})$$

with center $Y_k = R(1)(\underline{\mathrm{Max}}\ t_k)$.

Take for each λ *the corresponding transformation of basic objects with center* $Y_k \cap W_k^\lambda$

$$(\widetilde{W_k^\lambda}, (\mathfrak{a}_k^\lambda, b^\lambda), \widetilde{E_k^\lambda}) \overset{\pi_{k+1}^\lambda}{\longleftarrow} (\widetilde{W_{k+1}^\lambda}, (\mathfrak{a}_{k+1}^\lambda, b^\lambda), \widetilde{E_{k+1}^\lambda}).$$

Then

$$F_{k+1} := \cup \mathrm{Sing}(\mathfrak{a}_{k+1}^\lambda, b^\lambda)$$

is a closed subset of W_{k+1} *with*

$$F_{k+1} \cap W_{k+1}^\lambda = \mathrm{Sing}(\mathfrak{a}_{k+1}^\lambda, b^\lambda),$$

and the extended sequence belongs to $\mathfrak{C}$, *i.e.,*

$$(F_0, (W_0, E_0)) \leftarrow \cdots \leftarrow (F_k, (W_k, E_k)) \leftarrow (F_{k+1}, (W_{k+1}, E_{k+1})) \in \mathfrak{C}.$$

We have one of the following four cases:

A-1*:* $F_{k+1} = \emptyset$.

In this case, the extended sequence represents resolution of singularities of the general basic object $(\mathcal{F}_0, (W_0, E_0))$.

A-2: $F_{k+1} \neq \emptyset$ *and* $\max\ w\text{-ord}_{k+1} = 0$.

In this case, we can apply Corollary 4-9 to $(\mathcal{F}_{k+1}, (W_{k+1}, E_{k+1}))$ *and create a sequence of transformations representing resolution of singularities of* $(\mathcal{F}_{k+1}, (W_{k+1}, E_{k+1}))$. *Attaching this to the extended sequence, we obtain a sequence representing resolution of singularities of* $(\mathcal{F}_0, (W_0, E_0))$.

A-3: $F_{k+1} \neq \emptyset$, $\max\ w\text{-ord}_{k+1} > 0$, *and* $\max\ t_k > \max\ t_{k+1}$.

In this case, obviously the maximum of the t*-invariant drops.*

A-4: $F_{k+1} \neq \emptyset$, $\max\ w\text{-ord}_{k+1} > 0$, $\max\ t_k = \max\ t_{k+1}$, *and* $R(1)(\underline{\mathrm{Max}}\ t_{k+1}) = \emptyset$.

In this case, we go to Case B for the general basic object $(\mathcal{F}_{k+1}, (W_{k+1}, E_{k+1}))$.

Case B: $R(1)(\underline{\mathrm{Max}}\ t_k) = \emptyset$.

In this case, we construct a general basic object over $(G_k = \underline{\mathrm{Max}}\ t_k, (W_k, E''_k))$ *with a* $(d-1)$*-dimensional structure with the following property:*

We construct a sequence of transformations representing resolution of singularities of the general basic object $(\mathcal{G}_k, (W_k, E''_k))$ *by induction on the dimension of the structure*

$$(G_k, (W_k, E''_k)) \leftarrow \cdots \leftarrow (G_{k+N} = \emptyset, (W_{k+N}, E''_{k+N})).$$

We have the corresponding sequence of transformations (with the same centers)

$$(F_k, (W_k, E_k)) \leftarrow \cdots \leftarrow (F_{k+N}, (W_{k+N}, E_{k+N}))$$

which belongs to the collection $\mathfrak{C}$ *represented by the original general basic object* $(\mathcal{F}_0, (W_0, E_0))$ *when attached to the original sequence of transformations*

$$(F_0, (W_0, E_0)) \leftarrow \cdots \leftarrow (F_k, (W_k, E_k)) \leftarrow \cdots \leftarrow (F_{k+N}, (W_{k+N}, E_{k+N}))$$

such that it satisfies the conditions

(i) *for* $j = 1, ..., N$, *we have*

$(\heartsuit')$ $Y_{k+j-1} \subset \underline{\mathrm{Max}}\ t_{k+j-1} \subset \underline{\mathrm{Max}}\ w\text{-ord}_{k+j-1} \subset F_{k+j-1}$ *where* $\max\ w\text{-ord}_{k+j-1} > 0$,

(Recall that we have condition

$$(\heartsuit')\ Y_{i-1} \subset \underline{\mathrm{Max}}\ t_{i-1} \subset \underline{\mathrm{Max}}\ w\text{-ord}_{i-1} \text{ for } i = 1, ..., k$$

where $\max\ w\text{-ord}_{i-1} \geq \max\ w\text{-ord}_k > 0$ *by the case assumption.*

Note also that the condition (iii) below implies

$$\max\ w\text{-ord}_k = \cdots = \max\ w\text{-ord}_{k+N-1} > 0.)$$

(ii) $\underline{\mathrm{Max}}\ t_{k+j-1} = G_{k+j-1}$ *for* $j = 1, ..., N$,

(iii) $\max\ t_k = \cdots = \max\ t_{k+N-1}$

(Note that the invariants t_{k+j-1} *and* $w\text{-ord}_{k+j-1}$ *for* $j = 1, ..., N$ *are the ones defined for the general basic objects* $(\mathcal{F}_{k+j-1}, (W_{k+j-1}, E_{k+j-1}))$ *with the* d*-dimensional structures.)*

and that we have one of the following three cases:

B-1: $F_{k+N} = \emptyset$.

In this case, the extended sequence represents resolution of singularities of the general basic object $(\mathcal{F}_0, (W_0, E_0))$.

B-2: $F_{k+N} \neq \emptyset$ *and* $\max\ w\text{-ord}_{k+N} = 0$.

In this case, we can apply Corollary 4-9 to $(\mathcal{F}_{k+N}, (W_{k+N}, E_{k+N}))$ *and create a sequence of transformations representing resolution of singularities of* $(\mathcal{F}_{k+N}, (W_{k+N}, E_{k+N}))$. *Attaching this to the extended sequence, we obtain a sequence representing resolution of singularities of* $(\mathcal{F}_0, (W_0, E_0))$.

B-3: $F_{k+N} \neq \emptyset$, $\max\ w\text{-ord}_{k+N} > 0$, *and* $\max\ t_k > \max\ t_{k+N}$.

In this case, obviously the maximum of the t-invariant drops.

Since the set of values of the t-invariant satisfies the descending chain condition, after finitely many executions of the process described as above, we obtain the uniquely determined sequence of transformations representing resolution of singularities of $(\mathcal{F}_0, (W_0, E_0))$ *via the inductive algorithm.*

Remark 5-2.

(i) In **Case B**, we first construct such a general basic object $(\mathcal{G}_k, (W_k, E''_k))$ over $(G_k, (W_k, E''_k))$ with a d-dimensional structure $\{(\widetilde{W^\lambda_k}, (\mathfrak{b}^\lambda_k, e^\lambda), \widetilde{D^\lambda})\}$, where the basic objects in the charts are "simple", that satisfies the requirements specified in **Case B**. Then we find an open covering of each $\widetilde{W^\lambda_k}$ together with smooth hypersurfaces satisfying conditions 1 and 2 as described in the key inductive lemma (Lemma 3-1). This is done via the power of the t-invariant. Now the key inductive lemma implies that the general basic object $(\mathcal{G}_k, (W_k, E''_k))$ has a $(d-1)$-dimensional structure, and hence completes the inductive step of the algorithm above.

(ii) In **Case B**, the extended sequence

$$(F_0, (W_0, E_0)) \leftarrow \cdots \leftarrow (F_k, (W_k, E_k)) \leftarrow \cdots \leftarrow (F_{k+j-1}, (W_{k+j-1}, E_{k+j-1})),$$

for $j = 1, ..., N$, remains in **Case B** (cf. the construction described in (i), Giraud's Lemma (Claim 3-4), and requirements (i) (ii) for **Case B**). The general basic object over $(\underline{\text{Max}}\ t_{k+j}, (W_{k+j}, E''_{k+j}))$ we construct for $(\mathcal{F}_{k+j}, (W_{k+j}, E_{k+j}))$ under the prescription of **Case B** coincides with the general basic object $(\mathcal{G}_{k+j}, (W_{k+j}, E''_{k+j}))$ induced from $(\mathcal{G}_k, (W_k, E''_k))$ via the sequence

$$(G_k, (W_k, E''_k)) \leftarrow \cdots \leftarrow (G_{k+j}, (W_{k+j}, E''_{k+j})).$$

(iii) The process of extending the sequence

$$(F_0, (W_0, E_0)) \leftarrow \cdots \leftarrow (F_k, (W_k, E_k))$$

prescribed by the inductive algorithm, in order to obtain resolution of singularities of $(\mathcal{F}_0, (W_0, E_0))$, may seem depend on the number d, which specifies the dimension of the structure. However, it is not difficult to see that the process is actually

independent of d and that it is purely determined by the sequence of general basic objects

$$(\mathcal{F}_0,(W_0,E_0)) \leftarrow \cdots \leftarrow (\mathcal{F}_k,(W_k,E_k)).$$

In fact, if we are in possibility **P1**, then we are done and have nothing more to do. The process in possibility **P2** is independent of d, since it only depends on the invariant Γ_k, which is determined purely in terms of the collection $\mathfrak{C}_k$ represented by the general basic object $(\mathcal{F}_k,(W_k,E_k))$ (cf. Definition-Proposition 4-5). So suppose we are in possibility **P3**. Suppose that the general basic objects have d-dimensional structures and that at the k-th stage after l-repetitions of the processes in **Case B** we reach a general basic object with a $(d-l)$-dimensional structure, for which we are either in possibility **P2** or in **Case A** with center $Y_k \subset F_k$. If the same general basic objects have d'-dimensional structures (with $d' \geq d$), then what happens at the k-th stage is that after $(d'-d)+l$-repetitions of the processes in **Case B** we reach a general basic object with a $(d-l) = (d'-(d'-d+l))$-dimensional structure, for which we are either in possibility **P2** or in **Case A** with the same center $Y_k \subset F_k$. Therefore, we conclude that the dimension of the structures of the general basic objects has no effect on the process.

Proof of Theorem 5-1.

First we check the assertions for **P1**, **P2**, and **Case A** under **P3**, in the process prescribed by the inductive algorithm.

P1: Under this possibility, the sequence represents resolution of singularities of the general basic object $(\mathcal{F}_0,(W_0,E_0))$ and we are done.

P2: Under this possibility, we can apply Corollary 4-9 to $(\mathcal{F}_k,(W_k,E_k))$ and create a sequence of transformations representing resolution of singularities of $(\mathcal{F}_k,(W_k,E_k))$. Attaching this to the original sequence, we obtain a sequence representing resolution of singularities of $(\mathcal{F}_0,(W_0,E_0))$ and we are done.

P3: So we may assume in the following that we are under possibility **P3**.

Case A: $R(1)(\underline{\mathrm{Max}}\ t_k) \neq \emptyset$.

Let $p \in R(1)(\underline{\mathrm{Max}}\ t_k)$ be an arbitrary point.
Then there exists W_k^λ such that

$$p \in R(1)(\underline{\mathrm{Max}}\ t_k) \cap W_k^\lambda \subset \underline{\mathrm{Max}}\ t_k \cap W_k^\lambda \subset \underline{\mathrm{Max}}\ w\text{-ord}_k \cap W_k^\lambda = \underline{\mathrm{Max}}\ w\text{-ord}_k^\lambda.$$

Set $c_k^\lambda = b^\lambda \cdot \max\ w\text{-ord}_k^\lambda$.

Then since $\underline{\mathrm{Max}}\ w\text{-ord}_k^\lambda = V(\Delta^{c_k^\lambda - 1}(\overline{\mathfrak{a}_k^\lambda})) \subset \widetilde{W_k^\lambda}$ and since $\nu_p(\Delta^{c_k^\lambda-1}(\overline{\mathfrak{a}_k^\lambda})) = 1$ (cf. Lemma 1-4), there exists an open subset $p \in U_p \subset \widetilde{W_k^\lambda}$ and a regular parameter f_p defined over U_p such that $\underline{\mathrm{Max}}\ w\text{-ord}_k^\lambda \cap U_p \subset \{f_p = 0\} \subset U_p$ where $\{f_p = 0\}$ is a nonsingular closed subvariety of codimension one in U_p.

We see by shrinking U_p if necessary that this implies

$$p \in R(1)(\underline{\mathrm{Max}}\ t_k) \cap U_p = \{f_p = 0\} = \underline{\mathrm{Max}}\ w\text{-ord}_k \cap U_p.$$

Since $p \in R(1)(\underline{\mathrm{Max}}\ t_k)$ is arbitrary, we conclude that $R(1)(\underline{\mathrm{Max}}\ t_k)$ ($\subset \underline{\mathrm{Max}}\ t_k \subset \underline{\mathrm{Max}}\ w\text{-ord}_k \subset F_k$) is smooth, open and closed in $\underline{\mathrm{Max}}\ w\text{-ord}_k$ (i.e., a union of smooth connected components of $\underline{\mathrm{Max}}\ w\text{-ord}_k$ disjoint from each other).

Since obviously $R(1)(\underline{\text{Max}}\ t_k) \cap W_k^\lambda \subset \widetilde{W_k^\lambda}$ and $R(1)(\underline{\text{Max}}\ t_k) \cap W_k^\lambda \subset F_k \cap W_k^\lambda = \text{Sing}(\mathfrak{a}_k^\lambda, b^\lambda)$, we have only to show that $R(1)(\underline{\text{Max}}\ t_k) \cap W_k^\lambda$ is permissible with respect to $\widetilde{E_k^\lambda} = E_k \cap \widetilde{W_k^\lambda}$.

Observe first that

$$\max\ t_k = (\max\ w\text{-ord}_k, 0) \text{ or } (\max\ w\text{-ord}_k, 1),$$

since if the second factor is ≥ 2, then $\underline{\text{Max}}\ t_k$ has codimension at least two in $\widetilde{W_k^\lambda}$, which is against the case assumption of $R(1)(\underline{\text{Max}}\ t_k) \neq \emptyset$.

Suppose $\max\ t_k = (\max\ w\text{-ord}_k, 1)$. Then for any point $p \in R(1)(\underline{\text{Max}}\ t_k) \cap W_k^\lambda$ there exists an open neighborhood $U_p \subset \widetilde{W_k^\lambda}$ and $p \in \widetilde{H_j} \in E_k^- \cap \widetilde{W_k^\lambda} \subset \widetilde{E_k^\lambda}$ (See Definition-Proposition 4-5 for the definition of E_k^-.) such that

$$R(1)(\underline{\text{Max}}\ t_k) \cap U_p = \widetilde{H_j} \cap U_p.$$

Since $\widetilde{H_j}$ is clearly permissible with respect to $\widetilde{E_k^\lambda}$, we verify the permissibility in this case.

Suppose $\max\ t_k = (\max\ w\text{-ord}_k, 0)$. Then, by definition, no point $p \in R(1)(\underline{\text{Max}}\ t_k) \cap W_k^\lambda$ is contained in $E_k^- \cap W_k^\lambda$. (Hence $R(1)(\underline{\text{Max}}\ t_k) \cap W_k^\lambda$ is disjoint from $E_k^- \cap \widetilde{W_k^\lambda} = \widetilde{E_k^\lambda}^-$.) Thus we have only to check $R(1)(\underline{\text{Max}}\ t_k) \cap W_k^\lambda$ is permissible with respect to $E_k^+ \cap \widetilde{W_k^\lambda} = \widetilde{E_k^\lambda}^+$.

Let $D \subset R(1)(\underline{\text{Max}}\ t_k) \cap W_k^\lambda$ be an irreducible (and hence connected) component. Let k_o be the index such that

$$\max\ w\text{-ord}_{k_o - 1} > \max\ w\text{-ord}_{k_o} = \cdots = \max\ w\text{-ord}_k$$

as described in Definition-Proposition 4-5 (iii).

If D is one of the exceptional divisors for the morphism $\widetilde{W_{k_o}^\lambda} \leftarrow \widetilde{W_k^\lambda}$ (That is to say, if the strict transform of D is the pull-back of the irreducible component of the center for some transformation. Note that even in the case where we take the center to be a divisor (a subvariety of codimension one) and hence where the transformation is an isomorphism set-theoretically, we call D one of the *exceptional* divisors. See the convention of Definition 1-8 (iii).), then $D \subset \widetilde{E_k^\lambda}$ and hence is permissible with respect to $\widetilde{E_k^\lambda}^+$. (In fact, $D \subset \widetilde{E_k^\lambda}^+$ would imply that $w\text{-ord}_k(\eta_D) = 0$, where η_D is the generic point of D, which is against the case assumption $\max\ w\text{-ord}_k > 0$ of **<u>P3</u>**. So this case does not happen.)

If D is not any one of the exceptional divisors for the morphism $\widetilde{W_{k_o}^\lambda} \leftarrow \widetilde{W_k^\lambda}$, then $D_i \subset \underline{\text{Max}}\ w\text{-ord}_i$, for $i = k_o, ..., k$, where D_i is the image of D on $\widetilde{W_i^\lambda}$. (See the above note for the meaning of the *exceptional* divisors.) Therefore, D_i is a smooth connected component of $\underline{\text{Max}}\ w\text{-ord}_i$, by the same argument at the beginning of the proof for **Case A**. Therefore, any irreducible component of the center $Y_i \subset \underline{\text{Max}}\ w\text{-ord}_i$ is either contained in D_i or disjoint from D_i, while Y_i is permissible with respect to $\widetilde{E_i^\lambda}$. Observe that D_{k_o} is clearly permissible with

respect to $\widetilde{E^\lambda_{k_o}}^+$, since $\widetilde{E^\lambda_{k_o}}^+ = \emptyset$. The above analysis of the centers inductively implies that D_i is permissible with respect to $\widetilde{E^\lambda_i}^+$ for $i = k_o, ..., k$.

This completes the proof that $R(1)(\underline{\text{Max}}\ t_k) \cap W^\lambda_k$ is permissible with respect to $\widetilde{E^\lambda_k} = E_k \cap \widetilde{W^\lambda_k}$.

Now for the transformation

$$(F_k, (W_k, E_k)) \stackrel{\pi_{k+1}}{\longleftarrow} (F_{k+1}, (W_{k+1}, E_{k+1}))$$

with center $Y_k = R(1)(\underline{\text{Max}}\ t_k)$, it is obvious, since $\max\ t_k \geq \max\ t_{k+1}$ (cf. Proposition 4-8), that we have the following four cases:

A-1: $F_{k+1} = \emptyset$.
A-2: $F_{k+1} \neq \emptyset$ and $\max\ w\text{-ord}_{k+1} = 0$.
A-3: $F_{k+1} \neq \emptyset$, $\max\ w\text{-ord}_{k+1} > 0$, and $\max\ t_k > \max\ t_{k+1}$.
A-4: $F_{k+1} \neq \emptyset$, $\max\ w\text{-ord}_{k+1} > 0$, and $\max\ t_k = \max\ t_{k+1}$.

The assertions for cases **A-1**, **A-2**, **A-3** are obvious.

So we have only to prove that in case **A-4** we have $R(1)(\underline{\text{Max}}\ t_{k+1}) = \emptyset$. In that case, for each λ, the corresponding transformation of basic objects

$$(\widetilde{W^\lambda_k}, (\mathfrak{a}^\lambda_k, b^\lambda), \widetilde{E^\lambda_k}) \stackrel{\pi^\lambda_{k+1}}{\longleftarrow} (\widetilde{W^\lambda_{k+1}}, (\mathfrak{a}^\lambda_{k+1}, b^\lambda), \widetilde{E^\lambda_{k+1}})$$

is an isomorphism between $\widetilde{W^\lambda_k}$ and $\widetilde{W^\lambda_{k+1}}$. Moreover, for each irreducible component $D \subset R(1)(\underline{\text{Max}}\ t_k) \cap W^\lambda_k$ we have $w\text{-ord}_{k+1}(\eta_D) = w\text{-ord}^\lambda_{k+1}(\eta_D) = 0$, since $D \subset \widetilde{E^\lambda_{k+1}}$, where η_D is the generic point of D. Since $\max\ w\text{-ord}_{k+1} > 0$ and since $\max\ t_k = \max\ t_{k+1}$, we have $\eta_D \notin \underline{\text{Max}}\ t_{k+1}$ and we also have $(\underline{\text{Max}}\ t_k) \cap W^\lambda_k \supset (\underline{\text{Max}}\ t_{k+1}) \cap W^\lambda_{k+1}$ (cf. Proposition 4-8). Therefore, we finally conclude that $R(1)(\underline{\text{Max}}\ t_{k+1}) = \emptyset$.

This completes the proof of the assertions in **Case A** under possibility **<u>P3</u>**.

The rest of the proof is dedicated to verifying the assertions in **Case B** under possibility **<u>P3</u>**.

Case B: $R(1)(\underline{\text{Max}}\ t_k) = \emptyset$.

In order to construct a general basic object over $(G_k = \underline{\text{Max}}\ t_k, (W_k, E''_k))$ with the specified properties, we prove the following two lemmas, which, given a sequence of transformations and smooth morphisms of basic objects, construct basic objects whose singular loci coincide with <u>Max</u> w-ord and <u>Max</u> t, respectively.

Lemma 5-3. *Let*

$$\begin{aligned} (W_0, (J_0, b), E_0) &\stackrel{\pi_1}{\longleftarrow} (W_1, (J_1, b), E_1) \stackrel{\pi_2}{\longleftarrow} \cdots \\ (W_{i-1}, (J_{i-1}, b), E_{i-1}) &\stackrel{\pi_i}{\longleftarrow} (W_i, (J_i, b), E_i) \\ \cdots \stackrel{\pi_{k-1}}{\longleftarrow} (W_{k-1}, (J_{k-1}, b), E_{k-1}) &\stackrel{\pi_k}{\longleftarrow} (W_k, (J_k, b), E_k) \end{aligned}$$

be a sequence of transformations and smooth morphisms of basic objects such that

$$\max\ w\text{-ord}_k > 0.$$

Then there exists a simple basic object $(W'_k = W_k, (J'_k, b'), E'_k)$ *whose singular locus (as well as the singular loci of the basic objects in the sequences of transformations and smooth morphisms starting from it) coincides with the locus* $\underline{\mathrm{Max}}\ w\text{-ord}_k$ *of* $(W_k, (J_k, b), E_k)$ *(as well as the loci* $\underline{\mathrm{Max}}\ w$*-ord of the basic objects in the extended sequences of transformations and smooth morphisms satisfying condition* $(\heartsuit)$*) in the sense precisely formulated as follows:*

(α) *With each sequence of transformations and smooth morphisms of basic objects starting from* $(W'_k, (J'_k, b'), E'_k)$

$$(W'_k, (J'_k, b'), E'_k) = (W'_k, ((J'_k)_0, b'), E'_k) \stackrel{\pi'_{k+1}}{\longleftarrow} (W'_{k+1}, ((J'_k)_1, b'), E'_{k+1}) \stackrel{\pi'_{k+2}}{\longleftarrow} \cdots$$
$$\cdots \stackrel{\pi'_{N-1}}{\longleftarrow} (W'_{N-1}, ((J'_k)_{N-1}, b'), E'_{k+N-1}) \stackrel{\pi'_{k+N}}{\longleftarrow} (W'_{k+N}, ((J'_k)_N, b'), E'_{k+N})$$

satisfying the condition

$$\mathrm{Sing}((J'_k)_j, b') \neq \emptyset \textit{ for } j = 0, ..., N-1,$$

there corresponds an extension of the original sequence of transformations and smooth morphisms, satisfying condition $(\heartsuit)$ *for* $i = k + j + 1 \ \ (j = 0, ..., N-1)$*,*

$$\begin{aligned} &(W_0, (J_0, b), E_0) \stackrel{\pi_1}{\longleftarrow} \cdots \stackrel{\pi_{k-1}}{\longleftarrow} (W_{k-1}, (J_{k-1}, b), E_{k-1}) \stackrel{\pi_k}{\longleftarrow} \\ &(W_k, (J_k, b), E_k) \stackrel{\pi_{k+1}}{\longleftarrow} (W_{k+1}, (J_{k+1}, b), E_{k+1}) \stackrel{\pi_{k+2}}{\longleftarrow} \cdots \\ &\cdots \stackrel{\pi_{k+N-1}}{\longleftarrow} (W_{k+N-1}, (J_{k+N-1}, b), E_{k+N-1}) \stackrel{\pi_{k+N}}{\longleftarrow} (W_{k+N}, (J_{k+N}, b), E_{k+N}) \end{aligned}$$

such that the following conditions are satisfied:

(i) π'_{k+j+1} *and* π_{k+j+1} *are the transformations with the same centers or the same smooth morphisms (as abstract varieties) for* $j = 0, ..., N-1$ *with* $W'_{k+j+1} = W_{k+j+1}$ *(which means, in particular, that if* π'_{k+j+1} *is the transformation with center* $Y'_{k+j} \subset W'_{k+j}$ *which is permissible for* $(W'_{k+j}, ((J'_k)_j, b'), E'_{k+j})$*, then* π_{k+j+1} *is the transformation with the same center* $Y'_{k+j} \subset W'_{k+j} = W_{k+j}$ *which is also permissible for* $(W_{k+j}, (J_{k+j}, b), E_{k+j})$*),*

(ii) we have

either

$$\begin{cases} \max\ w\text{-ord}_k = \max\ w\text{-ord}_{k+1} = \cdots = \max\ w\text{-ord}_{k+N}, \ \textit{and} \\ \mathrm{Sing}((J'_k)_j, b') = \underline{\mathrm{Max}}\ w\text{-ord}_{k+j} \textit{ for } j = 0, ..., N \end{cases}$$

or

$$\begin{cases} \max\ w\text{-ord}_k = \max\ w\text{-ord}_{k+1} = \cdots = \max\ w\text{-ord}_{k+N-1} > \max\ w\text{-ord}_{k+N} \\ (\textit{or } \max\ w\text{-ord}_k = \max\ w\text{-ord}_{k+1} = \cdots = \max\ w\text{-ord}_{k+N-1} \ \ \& \ \ \mathrm{Sing}(J_{k+N}, b) = \emptyset), \ \textit{and} \\ \mathrm{Sing}((J'_k)_j, b') = \underline{\mathrm{Max}}\ w\text{-ord}_{k+j} \textit{ for } j = 0, ..., N-1 \ \ \& \ \ \mathrm{Sing}((J'_k)_N, b') = \emptyset. \end{cases}$$

(β) *Conversely, with each extension of the original sequence of transformations and smooth morphisms*

$$
\begin{array}{l}
(W_0,(J_0,b),E_0) \stackrel{\pi_1}{\longleftarrow} \cdots \stackrel{\pi_{k-1}}{\longleftarrow} (W_{k-1},(J_{k-1},b),E_{k-1}) \stackrel{\pi_k}{\longleftarrow} \\
(W_k,(J_k,b),E_k) \stackrel{\pi_{k+1}}{\longleftarrow} (W_{k+1},(J_{k+1},b),E_{k+1}) \stackrel{\pi_{k+2}}{\longleftarrow} \cdots \\
\cdots \stackrel{\pi_{k+N-1}}{\longleftarrow} (W_{k+N-1},(J_{k+N-1},b),E_{k+N-1}) \stackrel{\pi_{k+N}}{\longleftarrow} (W_{k+N},(J_{k+N},b),E_{k+N}),
\end{array}
$$

satisfying condition ($\heartsuit$) *for* $i = k+j+1 \quad (j = 0, ..., N-1)$ *and the condition*

$$\max\ w\text{-ord}_k = \max\ w\text{-ord}_{k+1} = \cdots = \max\ w\text{-ord}_{k+N-1},$$

there corresponds a sequence of transformations and smooth morphisms of basic objects starting from $(W'_k,(J'_k,b'),E'_k)$

$$
\begin{array}{l}
(W'_k,(J'_k,b'),E'_k) = (W'_k,((J'_k)_0,b'),E'_k) \stackrel{\pi'_{k+1}}{\longleftarrow} (W'_{k+1},((J'_k)_1,b'),E'_{k+1}) \stackrel{\pi'_{k+2}}{\longleftarrow} \cdots \\
\cdots \stackrel{\pi'_{N-1}}{\longleftarrow} (W'_{N-1},((J'_k)_{N-1},b'),E'_{k+N-1}) \stackrel{\pi'_{k+N}}{\longleftarrow} (W'_{k+N},((J'_k)_N,b'),E'_{k+N})
\end{array}
$$

satisfying the condition

$$\operatorname{Sing}((J'_k)_j,b') \neq \emptyset \textit{ for } j = 0,...,N-1$$

and conditions (i) and (ii) as in (α).

Proof.

Recall the characterization of the ideal $\overline{J_k}$ (cf. Definition 1-10 (ii))

$$J_k = I(H_{r+1})^{a_{r+1}} \cdots I(H_{r+k})^{a_{r+k}} \cdot \overline{J_k}.$$

We set

$$b_k = b \cdot \max\ w\text{-ord}_k.$$

We define the basic object $(W'_k,(J'_k,b'),E'_k)$ in the following way:

$$
\begin{cases}
W'_k = W_k \\
J'_k = \begin{cases} \overline{J_k} & \text{if } b_k \geq b \\ \overline{J_k}^{b-b_k} + \{I(H_{r+1})^{a_{r+1}} \cdots I(H_{r+k})^{a_{r+k}}\}^{b_k} & \text{if } b_k < b \end{cases} \\
b' = \begin{cases} b_k & \text{if } b_k \geq b \\ b_k(b-b_k) & \text{if } b_k < b \end{cases} \\
E'_k = E_k.
\end{cases}
$$

We check that the basic object $(W'_k,(J'_k,b'),E'_k)$ has the required properties.

Property (α)

$\boxed{\text{Case} : b_k \geq b}$

In this case, we observe

$$\begin{aligned} \xi_k \in \mathrm{Sing}(J'_k, b') &\Longleftrightarrow \xi_k \in \mathrm{Sing}(\overline{J_k}, b_k) \\ &\Longleftrightarrow \nu_{\xi_k}(\overline{J_k}) \geq b_k \\ &\Longleftrightarrow \nu_{\xi_k}(\overline{J_k}) \geq b_k \quad \& \quad \nu_{\xi_k}(J_k) \geq b \\ &\Longleftrightarrow \nu_{\xi_k}(\overline{J_k}) = b_k \quad \& \quad \nu_{\xi_k}(J_k) \geq b \\ &\Longleftrightarrow \xi_k \in \underline{\mathrm{Max}}\ w\text{-ord}_k, \end{aligned}$$

which implies

$$\mathrm{Sing}(J'_k, b') = \underline{\mathrm{Max}}\ w\text{-ord}_k.$$

Moreover, the equivalence of the conditions above also shows

$$\xi_k \in \mathrm{Sing}(J'_k, b') \Longrightarrow \nu_{\xi_k}(J'_k) = \nu_{\xi_k}(\overline{J_k}) = b_k = b',$$

verifying that $(W'_k, (J'_k, b'), E'_k)$ is a simple basic object.

Inductively, we can construct an extension of the original sequence (of the transformations with the same centers and the same smooth morphisms) such that for $j = 0, ..., N-1$, since $\mathrm{Sing}((J'_k)_j, b') \neq \emptyset$, we have

$$\begin{cases} (J'_k)_j = \overline{J_{k+j}} \\ \max\ w\text{-ord}_k = \cdots = \max\ w\text{-ord}_{k+j} \text{ i.e., } b_k = \cdots = b_{k+j} \geq b \\ \text{and hence } (J'_k)_j = J'_{k+j}. \end{cases}$$

Therefore, by the argument at the beginning applied to $(W'_{k+j}, ((J'_k)_j, b'), E'_{k+j}) = (W'_{k+j}, (J'_{k+j}, b'), E'_{k+j})$, we conclude

$$\mathrm{Sing}((J'_k)_j, b') = \mathrm{Sing}(J'_{k+j}, b') = \underline{\mathrm{Max}}\ w\text{-ord}_{k+j} \text{ for } j = 0, ..., N-1,$$

which also implies condition (i) as $E'_{k+j} = E_{k+j}$.

In the case $\mathrm{Sing}((J'_k)_N, b') \neq \emptyset$, we have

$$\begin{cases} (J'_k)_N = \overline{J_{k+N}} \\ \max\ w\text{-ord}_k = \cdots = \max\ w\text{-ord}_{k+N} \text{ i.e., } b_k = \cdots = b_{k+N} \geq b \\ \text{and hence } (J'_k)_N = J'_{k+N}, \end{cases}$$

which implies

$$\mathrm{Sing}((J'_k)_N, b') = \underline{\mathrm{Max}}\ w\text{-ord}_{k+N}.$$

Thus we are in the former case stated in condition (ii).

In the case $\mathrm{Sing}((J'_k)_N, b') = \emptyset$, we have

$$\begin{cases} (J'_k)_N = \overline{J_{k+N}} \\ \max\ w\text{-ord}_k = \cdots = \max\ w\text{-ord}_{k+N-1} > \max\ w\text{-ord}_{k+N} \text{ i.e., } b_k = \cdots = b_{k+N-1} > b_{k+N} \\ (\text{or } \max\ w\text{-ord}_k = \cdots = \max\ w\text{-ord}_{k+N-1} \text{ i.e., } b_k = \cdots = b_{k+N-1} \quad \& \quad \mathrm{Sing}(J_{k+N}, b) = \emptyset). \end{cases}$$

Thus we are in the latter case stated in condition (ii).

$\boxed{\text{Case} : b_k < b}$

In this case, we observe (cf. Remark 1-2 (ii))

$$\begin{aligned}
&\xi_k \in \mathrm{Sing}(J'_k, b') \\
&\quad \Longleftrightarrow \xi_k \in \mathrm{Sing}(\overline{J_k}^{b-b_k} + \{I(H_{r+1})^{a_{r+1}} \cdots I(H_{r+k})^{a_{r+k}}\}^{b_k}, b_k(b-b_k)) \\
&\quad \Longleftrightarrow \nu_{\xi_k}(\overline{J_k}^{b-b_k}) \geq b_k(b-b_k) \;\;\&\;\; \nu_{\xi_k}(\{I(H_{r+1})^{a_{r+1}} \cdots I(H_{r+k})^{a_{r+k}}\}^{b_k}) \geq b_k(b-b_k) \\
&\quad \Longleftrightarrow \nu_{\xi_k}(\overline{J_k}) \geq b_k \;\;\&\;\; \nu_{\xi_k}(I(H_{r+1})^{a_{r+1}} \cdots I(H_{r+k})^{a_{r+k}}) \geq b - b_k \\
&\quad \Longleftrightarrow \nu_{\xi_k}(I(H_{r+1})^{a_{r+1}} \cdots I(H_{r+k})^{a_{r+k}} \cdot \overline{J_k}) \geq b_k + (b-b_k) = b, \\
&\qquad\quad \nu_{\xi_k}(\overline{J_k}) = b_k \;\;\&\;\; \nu_{\xi_k}(I(H_{r+1})^{a_{r+1}} \cdots I(H_{r+k})^{a_{r+k}}) \geq b - b_k \\
&\quad \Longleftrightarrow \xi_k \in \underline{\mathrm{Max}}\ w\text{-}\mathrm{ord}_k,
\end{aligned}$$

which implies

$$\mathrm{Sing}(J'_k, b') = \underline{\mathrm{Max}}\ w\text{-}\mathrm{ord}_k.$$

Moreover, the equivalence of the conditions above also shows

$$\xi_k \in \mathrm{Sing}(J'_k, b') \Longrightarrow \nu_{\xi_k}(J'_k) = \nu_{\xi_k}(\overline{J_k}^{b-b_k}) = b_k(b-b_k) = b',$$

verifying that $(W'_k, (J'_k, b'), E'_k)$ is a simple basic object.

Inductively, we can construct an extension of the original sequence (of the transformations with the same centers and the same smooth morphisms) such that, since $\mathrm{Sing}((J'_k)_j, b') \neq \emptyset$, for $j = 0, ..., N-1$

$$\begin{cases} (J'_k)_j = \overline{J_{k+j}}^{b-b_{k+j}} + \{I(H_{r+1})^{a_{r+1}} \cdots I(H_{r+k+j})^{a_{r+k+j}}\}^{b_{k+j}} \\ \max\ w\text{-}\mathrm{ord}_k = \cdots = \max\ w\text{-}\mathrm{ord}_{k+j} \text{ i.e., } b_k = \cdots = b_{k+j} < b \\ \text{and hence } (J'_k)_j = J'_{k+j}. \end{cases}$$

Therefore, by the argument at the beginning applied to $(W'_{k+j}, ((J'_k)_j, b'), E'_{k+j}) = (W'_{k+j}, (J'_{k+j}, b'), E'_{k+j})$, we conclude

$$\mathrm{Sing}((J'_k)_j, b') = \mathrm{Sing}(J'_{k+j}, b') = \underline{\mathrm{Max}}\ w\text{-}\mathrm{ord}_{k+j} \text{ for } j = 0, ..., N-1,$$

which also implies condition (i) as $E'_{k+j} = E_{k+j}$.

In the case $\mathrm{Sing}((J'_k)_N, b') \neq \emptyset$, we have

$$\begin{cases} (J'_k)_N = \overline{J_{k+N}}^{b-b_{k+N}} + \{I(H_{r+1})^{a_{r+1}} \cdots I(H_{r+k+N})^{a_{r+k+N}}\}^{b_{k+N}} \\ \max\ w\text{-}\mathrm{ord}_k = \cdots = \max\ w\text{-}\mathrm{ord}_{k+N} \text{ i.e., } b_k = \cdots = b_{k+N} < b \\ \text{and hence } (J'_k)_N = J'_{k+N}, \end{cases}$$

which implies

$$\mathrm{Sing}((J'_k)_N, b') = \underline{\mathrm{Max}}\ w\text{-}\mathrm{ord}_{k+N}.$$

Thus we are in the former case stated in condition (ii).

In the case $\mathrm{Sing}((J'_k)_N, b') = \emptyset$, we have

$$\begin{cases} (J'_k)_N = \overline{J_{k+N}}^{b-b_{k+N-1}} + \{I(H_{r+1})^{a_{r+1}} \cdots I(H_{r+k+N})^{a_{r+k+N}}\}^{b_{k+N-1}} \\ \max\ w\text{-}\mathrm{ord}_k = \cdots = \max\ w\text{-}\mathrm{ord}_{k+N-1} > \max\ w\text{-}\mathrm{ord}_{k+N} \text{ i.e., } b_k = \cdots = b_{k+N-1} > b_{k+N} \\ (\text{or } \max\ w\text{-}\mathrm{ord}_k = \cdots = \max\ w\text{-}\mathrm{ord}_{k+N-1} \text{ i.e., } b_k = \cdots = b_{k+N-1}\ \ \&\ \ \mathrm{Sing}(J_{k+N}, b) = \emptyset). \end{cases}$$

Thus we are in the latter case stated in condition (ii).

Property (β)

The argument for verification of property (β) is identical to the one for verification of property (α), and is left to the reader as an exercise.

This completes the proof for Lemma 5-3.

Lemma 5-4. *Let*

$$\begin{gathered} (W_0, (J_0, b), E_0) \stackrel{\pi_1}{\leftarrow} (W_1, (J_1, b), E_1) \stackrel{\pi_2}{\leftarrow} \cdots \\ (W_{i-1}, (J_{i-1}, b), E_{i-1}) \stackrel{\pi_i}{\leftarrow} (W_i, (J_i, b), E_i) \\ \cdots \stackrel{\pi_{k-1}}{\leftarrow} (W_{k-1}, (J_{k-1}, b), E_{k-1}) \stackrel{\pi_k}{\leftarrow} (W_k, (J_k, b), E_k) \end{gathered}$$

be a sequence of transformations and smooth morphisms of basic objects with condition ($\heartsuit$)

$$(\heartsuit) \quad \left\{ \begin{array}{l} Y_{i-1} \subset \underline{\mathrm{Max}}\ w\text{-}\mathrm{ord}_{i-1}(\subset \mathrm{Sing}(J_{i-1}, b)) \\ \textit{whenever } \pi_i \textit{ is a transformation with center } Y_{i-1} \end{array} \right\} \ \textit{for } i = 1, ..., k$$

such that

$$\max\ w\text{-}\mathrm{ord}_k > 0.$$

Then there exists a simple basic object $(W''_k = W_k, (J''_k, b''), E''_k)$ *whose singular locus (as well as the singular loci of the basic objects in the sequences of transformations and smooth morphisms starting from it) coincides with the locus* $\underline{\mathrm{Max}}\ t_k$ *of* $(W_k, (J_k, b), E_k)$ *(as well as the loci* $\underline{\mathrm{Max}}\ t$ *of the basic objects in the extended sequences of transformations and smooth morphisms satisfying condition* ($\heartsuit'$)) *in the sense precisely formulated as follows:*

(α) *With each sequence of transformations and smooth morphisms of basic objects starting with* $(W''_k, (J''_k, b''), E''_k)$

$$\begin{gathered} (W''_k, (J''_k, b''), E''_k) = (W''_k, ((J''_k)_0, b''), E''_k) \stackrel{\pi''_{k+1}}{\leftarrow} (W''_{k+1}, ((J''_k)_1, b'), E''_{k+1}) \stackrel{\pi''_{k+2}}{\leftarrow} \cdots \\ \cdots \stackrel{\pi''_{k+N-1}}{\leftarrow} (W''_{k+N-1}, ((J''_k)_{N-1}, b''), E''_{k+N-1}) \stackrel{\pi''_{k+N}}{\leftarrow} (W''_{k+N}, ((J''_k)_N, b''), E''_{k+N}) \end{gathered}$$

with the condition

$$\mathrm{Sing}((J''_k)_j, b'') \neq \emptyset \ \textit{for } j = 0, ..., N-1,$$

there corresponds an extension of the original sequence of transformations and smooth morphisms

$$\begin{aligned}&(W_0,(J_0,b),E_0) \overset{\pi_1}{\longleftarrow} \cdots \overset{\pi_{k-1}}{\longleftarrow} (W_{k-1},(J_{k-1},b),E_{k-1}) \overset{\pi_k}{\longleftarrow}\\ &(W_k,(J_k,b),E_k) \overset{\pi_{k+1}}{\longleftarrow} (W_{k+1},(J_{k+1},b),E_{k+1}) \overset{\pi_{k+2}}{\longleftarrow} \cdots\\ &\ldots \overset{\pi_{k+N-1}}{\longleftarrow} (W_{k+N-1},(J_{k+N-1},b),E_{k+N-1}) \overset{\pi_{k+N}}{\longleftarrow} (W_{k+N},(J_{k+N},b),E_{k+N})\end{aligned}$$

with condition

$$(\heartsuit') \quad \left.\begin{cases} Y_{i-1} \subset \underline{\mathrm{Max}}\ t_{i-1} \subset \underline{\mathrm{Max}}\ w\text{-}\mathrm{ord}_{i-1}(\subset \mathrm{Sing}(J_{i-1},b)) \\ \textit{whenever } \pi_i \textit{ is a transformation with center } Y_{i-1} \end{cases}\right\} \textit{ for } i=k+1,...,N$$

satisfying the following conditions:

(i) π''_{k+j+1} *and* π_{k+j+1} *are the transformations with the same centers or the same smooth morphisms (as abstract varieties) for* $j=0,...,N-1$ *with* $W''_{k+j+1} = W_{k+j+1}$ *(which means, in particular, if* π''_{k+j+1} *is the transformation with center* $Y''_{k+j} \subset W''_{k+j}$ *which is permissible for* $(W''_{k+j},((J''_k)_j,b''),E''_{k+j})$*, then* π_{k+j+1} *is the transformation with the same center* $Y''_{k+j} \subset W''_{k+j} = W_{k+j}$ *which is also permissible for* $(W_{k+j},(J_{k+j},b),E_{k+j})$*),*

(ii) we have

either

$$\begin{cases} \max\ t_k = \max\ t_{k+1} = \cdots = \max\ t_{k+N}, \textit{ and} \\ \mathrm{Sing}((J''_k)_j,b'') = \underline{\mathrm{Max}}\ t_{k+j} \textit{ for } j=0,...,N \end{cases}$$

or

$$\begin{cases} \max\ t_k = \max\ t_{k+1} = \cdots = \max\ t_{k+N-1} > \max\ t_{k+N} \\ (\textit{or } \max\ t_k = \max\ t_{k+1} = \cdots = \max\ t_{k+N-1} \ \ \&\ \ \mathrm{Sing}(J_{k+N},b)=\emptyset), \textit{ and} \\ \mathrm{Sing}((J''_k)_j,b'') = \underline{\mathrm{Max}}\ t_{k+j} \textit{ for } j=0,...,N-1 \ \ \&\ \ \mathrm{Sing}((J''_k)_N,b'')=\emptyset. \end{cases}$$

(β) *Conversely, with each extension of the original sequence of transformations and smooth morphisms*

$$\begin{aligned}&(W_0,(J_0,b),E_0) \overset{\pi_1}{\longleftarrow} \cdots \overset{\pi_{k-1}}{\longleftarrow} (W_{k-1},(J_{k-1},b),E_{k-1}) \overset{\pi_k}{\longleftarrow}\\ &(W_k,(J_k,b),E_k) \overset{\pi_{k+1}}{\longleftarrow} (W_{k+1},(J_{k+1},b),E_{k+1}) \overset{\pi_{k+2}}{\longleftarrow} \cdots\\ &\ldots \overset{\pi_{k+N-1}}{\longleftarrow} (W_{k+N-1},(J_{k+N-1},b),E_{k+N-1}) \overset{\pi_{k+N}}{\longleftarrow} (W_{k+N},(J_{k+N},b),E_{k+N})\end{aligned}$$

with condition

$$(\heartsuit') \quad \left.\begin{cases} Y_{i-1} \subset \underline{\mathrm{Max}}\ t_{i-1} \subset \underline{\mathrm{Max}}\ w\text{-}\mathrm{ord}_{i-1}(\subset \mathrm{Sing}(J_{i-1},b)) \\ \textit{whenever } \pi_i \textit{ is a transformation with center } Y_{i-1} \end{cases}\right\} \textit{ for } i=k+1,...,N$$

and the condition

$$\max\ t_k = \max\ t_{k+1} = \cdots = \max\ t_{k+N-1},$$

there corresponds a sequence of transformations and smooth morphisms of basic objects starting from $(W''_k, (J''_k, b''), E''_k)$

$$(W''_k, (J''_k, b''), E''_k) = (W''_k, ((J''_k)_0, b''), E''_k) \overset{\pi''_{k+1}}{\longleftarrow} (W''_{k+1}, ((J''_k)_1, b'), E''_{k+1}) \overset{\pi''_{k+2}}{\longleftarrow} \cdots$$
$$\cdots \overset{\pi''_{k+N-1}}{\longleftarrow} (W''_{k+N-1}, ((J''_k)_{N-1}, b''), E''_{k+N-1}) \overset{\pi''_{k+N}}{\longleftarrow} (W''_{k+N}, ((J''_k)_N, b''), E''_{k+N})$$

satisfying the condition

$$\mathrm{Sing}((J''_k)_j, b'') \neq \emptyset \textit{ for } j = 0, ..., N-1$$

and conditions (i) and (ii) as in (α).

Moreover, the basic object $(W''_k = W_k, (J''_k, b''), E''_k)$ *has an open covering* $\{(W''_k)^\gamma\}_{\gamma \in \Gamma}$ *satisfying conditions 1 and 2 of the key inductive lemma (Lemma 3-1): for each* $\gamma \in \Gamma$, *there exists a smooth hypersurface* $(W''_k)^\gamma_h \subset (W''_k)^\gamma$, *embedded as a closed subscheme, such that*

1. $I((W''_k)^\gamma_h) \subset \Delta^{b''-1}(J''_k)|_{(W''_k)^\gamma}$ *(and hence* $(W''_k)^\gamma_h \supset \mathrm{Sing}(J''_k, b'') \cap (W''_k)^\gamma$*), and*

2. $(W''_k)^\gamma_h$ *is permissible with respect to* $E''_k \cap (W''_k)^\gamma$, *and* $(W''_k)^\gamma_h$ *is not contained in* E''_k, *i.e.,* $(W''_k)^\gamma_h \not\subset E''_k$.

Proof.

Let $(W'_k, (J'_k, b'), E'_k)$ be the basic object we constructed as in Lemma 5.3.

Let

$$\max\ t_k = (\max\ w\text{-ord}_k, n).$$

We define the basic object $(W''_k, (J''_k, b'), E''_k)$ in the following way:

$$\begin{cases} W''_k = W_k \\ J''_k = J'_k + \displaystyle\prod_{\{H_{s_1},...,H_{s_n}\} \subset E^-_k \text{ with } H_{s_1},...,H_{s_n} \text{ distinct}} \{I(H_{s_1}) + \cdots + I(H_{s_n})\}^{b'} \\ b'' = b' \\ E''_k = E^+_k. \end{cases}$$

Recall that $E^-_k = \{H_1, ..., H_r, ..., H_{r+k_o}\}$ as a subset of $E_k = \{H_1, ..., H_r, ..., H_{r+k_o}, H_{r+k_o+1} ..., H_{r+k}\}$ and that E^+_k is the complement of E^-_k in E_k, where k_o is the index (See Definition 1-10 (iii).) so that

$$\max\ w\text{-ord}_{k_o-1} > \max\ w\text{-ord}_{k_o} = \cdots = \max\ w\text{-ord}_k.$$

Note that the order of the ideal $I(H_{s_1}) + \cdots + I(H_{s_n})$ is the characteristic function of the set $H_{s_1} \cap \cdots \cap H_{s_n}$, i.e.,

$$\nu_{\xi_k}(I(H_{s_1}) + \cdots + I(H_{s_n})) = \begin{cases} 0 & \text{if} \quad \xi_k \notin H_{s_1} \cap \cdots \cap H_{s_n} \\ 1 & \text{if} \quad \xi_k \in H_{s_1} \cap \cdots \cap H_{s_n}. \end{cases}$$

Note also that over a point $\xi_k \in \underline{\mathrm{Max}}\ w\text{-ord}_k$ NO two distinct intersections $H_{s_1} \cap \cdots \cap H_{s_n}$ (with $H_{s_1}, ..., H_{s_n}$ distinct) and $H_{s'_1} \cap \cdots \cap H_{s'_n}$ (with $H_{s'_1}, ..., H_{s'_n}$ distinct) can meet, since n is the maximum of the number of such divisors in E^-_k that intersect at a point in $\underline{\mathrm{Max}}\ w\text{-ord}_k$.

We check that the basic object $(W''_k, (J''_k, b'), E''_k)$ has the required properties.

Property (α)

We observe (cf. Remark 1-2 (ii))

$$
\begin{aligned}
&\xi_k \in \mathrm{Sing}(J''_k, b'') \\
&\Longleftrightarrow \xi_k \in \mathrm{Sing}(J'_k + \prod_{\{H_{s_1},\cdots,H_{s_n}\}\subset E^-_k \text{ with } H_{s_1},\ldots,H_{s_n} \text{ distinct}} \{I(H_{s_1}) + \cdots + I(H_{s_n})\}^{b'}, b') \\
&\Longleftrightarrow \nu_{\xi_k}(J'_k) \geq b' \quad \& \quad \nu_{\xi_k}(\prod_{\{H_{s_1},\cdots,H_{s_n}\}\subset E^-_k \text{ with } H_{s_1},\ldots,H_{s_n} \text{ distinct}} \{I(H_{s_1}) + \cdots + I(H_{s_n})\}^{b'}) \geq b' \\
&\Longleftrightarrow \nu_{\xi_k}(J'_k) = b' \\
&\qquad \& \quad \xi_k \in H_{s_1} \cap \cdots \cap H_{s_n} \text{ for some } \{H_{s_1},\cdots,H_{s_n}\} \subset E^-_k \text{ with } H_{s_1},\ldots,H_{s_n} \text{ distinct} \\
&\Longleftrightarrow \xi_k \in \underline{\mathrm{Max}}\ t_k,
\end{aligned}
$$

which implies

$$\mathrm{Sing}(J''_k, b'') = \underline{\mathrm{Max}}\ t_k.$$

Moreover, the equivalence of the conditions above also implies

$$\xi_k \in \mathrm{Sing}(J''_k, b'') \Longrightarrow \nu_{\xi_k}(J''_k) = \nu_{\xi_k}(J'_k) = b' = b'',$$

verifying that $(W''_k, (J''_k, b''), E''_k)$ is a simple basic object.

Inductively, we can construct an extension of the original sequence (of the transformations with the same centers and the same smooth morphisms) such that for $j = 0, \cdots, N-1$, since $\mathrm{Sing}((J''_k)_j, b'') \neq \emptyset$, we have

$$
\begin{cases}
(J''_k)_j = J'_{k+j} + \prod_{\{H_{s_1},\cdots,H_{s_n}\}\subset E^-_{k+j} \text{ with } H_{s_1},\ldots,H_{s_n} \text{ distinct}} \{I(H_{s_1}) + \cdots + I(H_{s_n})\}^{b'} \\
\max\ t_k = \cdots = \max\ t_{k+j} \\
\text{and hence } (J''_k)_j = J''_{k+j}.
\end{cases}
$$

Therefore, by the argument at the beginning applied to $(W''_{k+j}, ((J''_k)_j, b''), E''_{k+j}) = (W''_{k+j}, (J''_{k+j}, b''), E''_{k+j})$, we conclude

$$\mathrm{Sing}((J''_k)_j, b'') = \mathrm{Sing}(J''_{k+j}, b'') = \underline{\mathrm{Max}}\ t_{k+j} \text{ for } j = 0, \cdots, N-1.$$

Suppose that π''_{k+j+1} is the transformation with the center $Y''_{k+j} \subset W''_{k+j} = W_{k+j}$ permissible for $(W''_{k+j}, ((J''_k)_j = J''_{k+j}, b''), E''_{k+j})$ where $j = 0, \cdots, N-1$. Then remark that Y''_{k+j} is contained in $H_s \in E^-_{k+j}$ if $Y''_{k+j} \cap H_s \neq \emptyset$, since $Y''_{k+j} \subset \mathrm{Sing}((J''_k)_j, b'') = \underline{\mathrm{Max}}\ t_{k+j}$, and that Y''_{k+j} is permissible with respect to $E''_{k+j} = E^+_{k+j}$ by definition. This implies that the center $Y''_{k+j} \subset W''_{k+j} = W_{k+j}$ is permissible for $(W_{k+j}, (J_{k+j}, b), E_{k+j})$, verifying condition (i).

In the case $\mathrm{Sing}((J''_k)_N, b'') \neq \emptyset$, we have

$$
\begin{cases}
(J''_k)_N = J'_{k+N} + \prod_{\{H_{s_1},\cdots,H_{s_n}\}\subset E^-_{k+N} \text{ with } H_{s_1},\ldots,H_{s_n} \text{ distinct}} \{I(H_{s_1}) + \cdots + I(H_{s_n})\}^{b'} \\
\max\ t_k = \cdots = \max\ t_{k+N} \\
\text{and hence } (J''_k)_N = J''_{k+N},
\end{cases}
$$

which implies

$$\mathrm{Sing}((J''_k)_N, b'') = \underline{\mathrm{Max}}\ t_{k+N}.$$

Thus we are in the former case stated in condition (ii).

In the case $\mathrm{Sing}((J''_k)_N, b'') = \emptyset$, we have

$$\begin{cases} (J''_k)_N = (J'_k)_N + \prod_{\{H_{s_1},\cdots,H_{s_n}\}\subset E^-_{k+N-1}\subset E_{k+N} \text{ with } H_{s_1},\ldots,H_{s_n} \text{ distinct}} \{I(H_{s_1}) + \cdots + I(H_{s_n})\}^{b'} \\ \max\ t_k = \cdots = \max\ t_{k+N-1} > \max\ t_{k+N} \\ (\text{or } \max\ t_k = \cdots = \max\ t_{k+N-1}\ \ \&\ \ \mathrm{Sing}(J_{k+N}, b) = \emptyset). \end{cases}$$

Thus we are in the latter case stated in condition (ii). (Note in the last case of $\mathrm{Sing}((J''_k)_N, b'') = \emptyset$ that when the morphism $\pi''_{k+N} = \pi_{k+N}$ is a transformation the above notation $E^-_{k+N-1} \subset E_{k+N}$ denotes the set of the strict transforms in E_{k+N} of the divisors in E^-_{k+N-1} and that when the morphism $\pi''_{k+N} = \pi_{k+N}$ is a smooth morphism the above notation $E^-_{k+N-1} \subset E_{k+N}$ denotes the set of the inverse images in E_{k+N} of the divisors in E^-_{k+N-1}.)

Property (β)

The argument for verification of property (β) is identical to the one for verification of property (α), and is left to the reader as an exercise.

Now we show that $(W''_k, (J''_k, b''), E''_k)$ has an open covering $\{(W''_k)^\gamma\}_{\gamma\in\Gamma}$ with smooth hypersurfaces $(W''_k)^\gamma_h \subset (W''_k)^\gamma$, embedded as closed subschemes, satisfying conditions 1 and 2 of the key inductive lemma (cf. Lemma 3-1).

Let k_o be the index as before (See Definition 1-10 (iii).) so that

$$\max\ w\text{-ord}_{k_o-1} > \max\ w\text{-ord}_{k_o} = \cdots = \max\ w\text{-ord}_k.$$

Looking at the basic object $(W_{k_o}, (J_{k_o}, b), E_{k_o})$, we consider the basic object $(W'_{k_o} = W_{k_o}, (J'_{k_o}, b'), E'_{k_o} = E_{k_o})$ as constructed in Lemma 5-3.

First remark that by Lemma 5-3 the part after the index k_o of the original sequence of transformations and smooth morphisms of basic objects (satisfying condition ($\heartsuit$))

$$(W_{k_o}, (J_{k_o}, b), E_{k_o}) \leftarrow \cdots \leftarrow (W_k, (J_k, b), E_k)$$

gives rise to a sequence of transformations with the same centers and the same smooth morphisms (as abstract varieties) of basic objects starting from $(W'_{k_o}, (J'_{k_o}, b'), E'_{k_o})$

$$(W'_{k_o} = W_{k_o}, (J'_{k_o}, b'), E'_{k_o} = E_{k_o}) \leftarrow \cdots \leftarrow (W'_k = W_k, ((J'_{k_o})_{k-k_o}, b'), E'_k = E_k).$$

Secondly, since $(W'_{k_o}, (J'_{k_o}, b'), E'_{k_o})$ is simple (cf. Remark 1-5), there exists an open covering $\{(W'_{k_o})^\gamma\}_{\gamma\in\Gamma}$ satisfying the following condition: for each $\gamma \in \Gamma$, there exists a smooth hypersurface $(W'_{k_o})^\gamma_h$, embedded as a closed subscheme, such that

1. $I((W'_{k_o})^\gamma_h) \subset \Delta^{b'-1}(J'_{k_o})|_{(W'_{k_o})^\gamma}$.

Remark that we do not (and can not) require any transversality condition on $(W'_{k_o})^\gamma_h$ with respect to $E'_{k_o} = E_{k_o}$.

We claim that the open covering $\{(W''_k)^\gamma\}_{\gamma\in\Gamma}$ where $(W''_k)^\gamma$ are the inverse images $(W'_k)^\gamma$ of $(W'_{k_o})^\gamma$, together with $(W''_k)^\gamma_h \subset (W''_k)^\gamma$ where $(W''_k)^\gamma_h$ are the strict transforms $(W'_k)^\gamma_h$ of $(W'_{k_o})^\gamma_h$, satisfies conditions 1 and 2 of the key inductive lemma as required. (We note that when we have a smooth morphism we call the inverse image of the hypersurface "the strict transform" by abuse of language.)

In order to check condition 1, by the repetitive use of Giraud's Lemma (cf. Claim 3-4), we see that

$$I((W''_k)^\gamma_h) \subset \Delta^{b'-1}((J'_{k_o})_{k-k_o}).$$

Then noting that

$$(J'_{k_o})_{k-k_o} = J'_{k_o+(k-k_o)} = J'_k$$
$$J'_k \subset J''_k \text{ and hence } \Delta^{b'-1}(J'_k) \subset \Delta^{b'-1}(J''_k),$$

we finally conclude that

$$I((W''_k)^\gamma_h) \subset \Delta^{b'-1}(J''_k).$$

In order to check condition 2, $(W'_j)^\gamma_h$ being the strict transform of $(W'_{k_o})^\gamma_h$, inductively for $j = k_o+1, ..., k$, we see, whenever π'_j is the transformation with center $Y'_{j-1} \subset \mathrm{Sing}((J'_{k_o})_{j-1-k_o}, b') = \mathrm{Sing}(J'_{j-1}, b') = \underline{\mathrm{Max}}\ w\text{-ord}_{j-1}$, that

$$Y'_{j-1} \cap (W'_{j-1})^\gamma \subset (W'_{j-1})^\gamma_h \quad \text{(cf. Claim 3-4)}$$
$$Y'_{j-1} \text{ permissible with respect to } E'_{j-1} \cap (W'_{j-1})^\gamma = E_{j-1} \cap (W'_{j-1})^\gamma,$$
$$\text{(and hence } Y'_{j-1} \text{ permissible with respect to } E^+_{j-1} \cap (W'_{j-1})^\gamma),$$

which implies inductively (See the argument in **Case A** for permissibility of an irreducible component $D \subset R(1)(\underline{\mathrm{Max}}\ t_k \cap W^\lambda_k)$ with respect to $E^+_k \cap W^\lambda_k$ in the case D is not any one of the exceptional divisor $\widetilde{W^\lambda_{k_o}} \leftarrow \widetilde{W^\lambda_k}$.) that

for each $\gamma \in \Gamma$, $(W'_j)^\gamma_h$ is permissible with respect to $E^+_j \cap (W'_j)^\gamma$ and $(W'_j)^\gamma_h \not\subset E^+_j$.

In particular, for $j = k$, we have condition 2:

2. $(W''_k)^\gamma_h = (W'_k)^\gamma_h$ is permissible with respect to $E^+_k \cap (W''_k)^\gamma = E''_k \cap (W''_k)^\gamma$ and $(W''_k)^\gamma_h \not\subset E''_k$.

This completes the proof of Lemma 5-4.

Conclusion of the proof for the assertions in **Case B** *under possibility* **<u>P3</u>**

Now we go back to the proof of the assertions in **Case B**: $R(1)(\underline{\mathrm{Max}}\ t_k) = \emptyset$.

We construct a general basic object over $(G_k = \underline{\mathrm{Max}}\ t_k, (W_k, E''_k = E^+_k))$, with a d-dimensional structure first, by specifying its charts of basic objects $\{(\widetilde{W''^\lambda_k}, (\mathfrak{a}''^\lambda_k, b''^\lambda), \widetilde{E''^\lambda_k})\}$ of dimension d in the following way:

Let $\{(\widetilde{W^\lambda_k}, (\mathfrak{a}^\lambda_k, b^\lambda), \widetilde{E^\lambda_k})$ be the charts for the general basic objects $(\mathcal{F}_k, (W_k, E_k))$ arising from the sequence (cf. Note 4-3)

$$(F_0, (W_0, E_0)) \leftarrow \cdots \leftarrow (F_k, (W_k, E_k)).$$

We take $\widetilde{W_k''^{\lambda}} = \widetilde{W_k^{\lambda}}$.

If $\widetilde{W_k''^{\lambda}} \cap \underline{\mathrm{Max}}\ t_k = \emptyset$, then we take the basic object $(\widetilde{W_k''^{\lambda}}, (\mathfrak{a}_k''^{\lambda}, b''^{\lambda}), \widetilde{E_k''^{\lambda}})$ to be

$$\begin{cases} \widetilde{W_k''^{\lambda}} = \widetilde{W_k^{\lambda}} \\ \mathfrak{a}_k''^{\lambda} = \mathcal{O}_{\widetilde{W_k''^{\lambda}}} \\ b''^{\lambda} = 1 \\ \widetilde{E_k''^{\lambda}} = E_k^{+} \cap \widetilde{W_k''^{\lambda}}. \end{cases}$$

If $\widetilde{W_k''^{\lambda}} \cap \underline{\mathrm{Max}}\ t_k \neq \emptyset$, then we take the basic object $(\widetilde{W_k''^{\lambda}}, (\mathfrak{a}_k''^{\lambda}, b''^{\lambda}), \widetilde{E_k''^{\lambda}})$ to be

$$\begin{cases} \widetilde{W_k''^{\lambda}} = \widetilde{W_k^{\lambda}} \\ \mathfrak{a}_k''^{\lambda} = (\mathfrak{a}_k^{\lambda})'' \text{ as constructed in Lemma 5-4} \\ b''^{\lambda} = (b^{\lambda})'' \text{ as constructed in Lemma 5-4} \\ \widetilde{E_k''^{\lambda}} = \widetilde{E_k^{\lambda}}^{+} = E_k^{+} \cap \widetilde{W_k''^{\lambda}}. \end{cases}$$

Let $\mathfrak{C}_G$ be the collection of all the sequences of transformations and smooth morphisms of pairs with specified closed subsets, starting with $(G_k, (W_k, E_k''))$, which satisfy condition (GB-1) with respect to the charts $\{(\widetilde{W_k''^{\lambda}}, (\mathfrak{a}_k''^{\lambda}, b''^{\lambda}), \widetilde{E_k''^{\lambda}})\}$. Then condition (GB-3) is trivially satisfied by the construction, whereas condition (GB-0) is a consequence of the statements of Lemma 5-4 for $N = 0$ and condition (GB-2) a consequence of the statements of Lemma 5-4 for N general.

Therefore, the collection $\mathfrak{C}_G$ is represented by a general basic object $(\mathcal{G}_k, (W_k, E_k''))$ over $(G_k, (W_k, E_k''))$ with a d-dimensional structure, having charts $\{(\widetilde{W_k''^{\lambda}}, (\mathfrak{a}_k''^{\lambda}, b''^{\lambda}), \widetilde{E_k''^{\lambda}})\}$ of dimension d. (In short and roughly speaking, the general basic object $(\mathcal{G}_k, (W_k, E_k''))$ is the one whose specified closed subsets coincides with the loci $\underline{\mathrm{Max}}\ t$ of the sequences of transformations and smooth morphisms of general basic objects satisfying condition ($\heartsuit'$) and extending the original sequence

$$(\mathcal{F}_0, (W_0, E_0)) \leftarrow \cdots \leftarrow (\mathcal{F}_k, (W_k, E_k).)$$

Now the "Moreover" part of Lemma 5-4, applied to the charts $\{(\widetilde{W_k''^{\lambda}}, (\mathfrak{a}_k''^{\lambda}, b''^{\lambda}), \widetilde{E_k''^{\lambda}})\}$, and the key inducive lemma imply that the general basic object $(\mathcal{G}_k, (W_k, E_k''))$, which represents the collection $\mathfrak{C}_G$, has a $(d-1)$-dimensional structure.

It also follows from Lemma 5-4 that the general basic object $(\mathcal{G}_k, (W_k, E_k''))$ has the following two properties (α) and (β):

(α) With each sequence in $\mathfrak{C}_G$

$$(G_k, (W_k, E_k'')) \stackrel{\pi''_{k+1}}{\longleftarrow} \cdots \stackrel{\pi''_{k+N}}{\longleftarrow} (G_{k+N}, (W_{k+N}, E''_{k+N}))$$

satisfying the condition

$$G_{k+j} \neq \emptyset \text{ for } j = 0, ..., N-1,$$

there corresponds an extension of the original sequence of transformations and smooth morphisms of general basic objects

$$
\begin{aligned}
&(\mathcal{F}_0,(W_0,E_0)) \overset{\pi_1}{\leftarrow} \cdots \overset{\pi_{k-1}}{\leftarrow} (\mathcal{F}_{k-1},(W_{k-1},E_{k-1})) \overset{\pi_k}{\leftarrow} \\
&(\mathcal{F}_k,(W_k,E_k)) \overset{\pi_{k+1}}{\leftarrow} \cdots \overset{\pi_{k+N}}{\leftarrow} (\mathcal{F}_{k+N},(W_{k+N},E_{k+N}))
\end{aligned}
$$

with condition

$$
(\heartsuit') \quad \left\{ \begin{array}{l} Y_{i-1} \subset \underline{\mathrm{Max}}\ t_{i-1} \subset \underline{\mathrm{Max}}\ w\text{-ord}_{i-1} (\subset F_{i-1}) \\ \text{whenever } \pi_i \text{ is a transformation with center } Y_{i-1} \end{array} \right\} \text{ for } i = k+1, ..., N
$$

satisfying the following conditions:

(i) π''_{k+j+1} and π_{k+j+1} are the transformations with the same centers or the same smooth morphisms (as abstract varieties) for $j = 0, ..., N-1$ (which means, in particular, if π''_{k+j+1} is the transformation with center $Y''_{k+j} \subset W_{k+j}$ which is permissible for $(G_{k+j},(W_{k+j},E''_{k+j}))$, then π_{k+j+1} is the transformation with the same center $Y''_{k+j} \subset W_{k+j} = W''_{k+j}$ which is also permissible for $(F_{k+j},(W_{k+j},E_{k+j}))$,

(ii) we have

either

$$
\left\{ \begin{array}{l} \max\ t_k = \max\ t_{k+1} = \cdots = \max\ t_{k+N}, \text{ and} \\ G_{k+j} = \underline{\mathrm{Max}}\ t_{k+j} \text{ for } j = 0, \cdots, N \end{array} \right.
$$

or

$$
\left\{ \begin{array}{l} \max\ t_k = \max\ t_{k+1} = \cdots = \max\ t_{k+N-1} > \max\ t_{k+N} \\ (\text{or } \max\ t_k = \max\ t_{k+1} = \cdots = \max\ t_{k+N-1} \quad \& \quad F_{k+N} = \emptyset), \text{ and} \\ G_{k+j} = \underline{\mathrm{Max}}\ t_{k+j} \text{ for } j = 0, \cdots, N-1 \quad \& \quad G_{k+N} = \emptyset. \end{array} \right.
$$

(β) Conversely, with each extension of the original sequence of transformations and smooth morphisms of general basic objects

$$
\begin{aligned}
&(\mathcal{F}_0,(W_0,E_0)) \overset{\pi_1}{\leftarrow} \cdots \overset{\pi_{k-1}}{\leftarrow} (\mathcal{F}_{k-1},(W_{k-1},E_{k-1})) \overset{\pi_k}{\leftarrow} \\
&(\mathcal{F}_k,(W_k,E_k)) \overset{\pi_{k+1}}{\leftarrow} \cdots \overset{\pi_{k+N}}{\leftarrow} (\mathcal{F}_{k+N},(W_{k+N},E_{k+N}))
\end{aligned}
$$

with condition

$$
(\heartsuit') \quad \left\{ \begin{array}{l} Y_{i-1} \subset \underline{\mathrm{Max}}\ t_{i-1} \subset \underline{\mathrm{Max}}\ w\text{-ord}_{i-1} (\subset F_{i-1}) \\ \text{whenever } \pi_i \text{ is a transformation with center } Y_{i-1} \end{array} \right\} \text{ for } i = k+1, ..., N
$$

and the condition

$$
\max\ t_k = \max\ t_{k+1} = \cdots = \max\ t_{k+N-1},
$$

there corresponds a sequence in $\mathfrak{C}_G$

$$
(G_k,(W_k,E''_k)) \overset{\pi''_{k+1}}{\leftarrow} \cdots \overset{\pi''_{k+N}}{\leftarrow} (G_{k+N},(W_{k+N},E''_{k+N}))
$$

satisfying the condition

$$G_{k+j} \neq \emptyset \text{ for } j = 0, ..., N-1$$

and conditions (i) and (ii) as in (α).

(Note that the above properties (α) and (β) provide a characterization of the general basic object $(\mathcal{G}_k, (W_k, E''_k))$ free of the description using the charts we construct, and that it is via this property of the collections $\mathfrak{C}$ and $\mathfrak{C}_G$ that we verify $\mathfrak{C}_G$ satisfies conditions (GB-0) and (GB-2).)

The assertions in **B-1**, **B-2**, **B-3** follow immediately from this.

Starting from $(F_0, (W_0, E_0))$, the inductive algorithm allows one by induction on the dimension d of the structure to construct a unique extension of the sequence of transformations (constructed up to the k-th stage) satisfying condition ($\heartsuit'$) such that

either $F_{k+N} = \emptyset$

where resolution of singularities is already achieved,

or $F_{k+N} \neq \emptyset$ & max $w\text{-ord}_{k+N} = 0$

where reslotuion of singularities is reduced to that of a monomial case,

or $F_{k+N} \neq \emptyset, \max\ w\text{-ord}_{k+N} > 0$ & $\emptyset \max t_k > \max\ t_{k+N}$.

Thanks to condition (GB-3), the values of the t-inavariant are in $\frac{1}{c!}\mathbb{Z}_{\geq 0} \times \mathbb{Z}_{\geq 0}$ and hence satisfy the descending chain condition. Therefore, the third possibility can not happen infinitely many times. Thus after finitely many executions of the inductive algorithm, we obtain the sequence representing resolution of singularities as asserted.

We note that resolution of singularities of a general basic object with a 1-dimensional structure is obvious. In fact, at the stage $i = 0$, we are always in **Case A** only with the possibilities **A-1**, **A-2** at the stage $i = 1$. If **A-1** is the case, then resolution of singularities is already achieved. If **A-2** is the case, then we are reduced to the monomial case, where resolution of singularities is achieved by nothing but a sequence of consecutive point blowups. This supports the tower of the inductional algorithm at the bottom $d = 1$.

This completes the proof of Theorem 5-1.

Remark 5-5.

The role of the invariant t is absolutely crucial at a couple of places, in the inductive algorithm presented as Theorem 5-1, e.g.:

- permissiblity of the center $R(1)(\underline{\text{Max}}\ t_k)$ in **Case A**.
- the proof that $(W''_k, (J''_k, b''), E''_k)$ satisfies conditions 1 and 2 as stated in the key inductive lemma.

We refer the reader to Chapter 6 for a more and natural explanation of the origin of the t-invariant by breaking up the inductive algorithm into a couple of natural reduction steps.

Corollary 5-6 (Resolution of singularities of a basic object). *Let* $(W_0, (J_0, b), E_0)$ *be a basic object with* $d = \dim W_0$*. Then there exists a sequence of transformations only*

$$(W_0, (J_0, b), E_0) \leftarrow \cdots \leftarrow (W_k, (J_k, b), E_k)$$

satisfying the following condition $(\heartsuit')$ *for* $i = 1, ..., k$

$$(\heartsuit') \quad Y_{i-1} \subset \underline{\mathrm{Max}}\ t_{i-1} \subset \underline{\mathrm{Max}}\ w\text{-}\mathrm{ord}_{i-1} \ \textit{if}\ \max\ w\text{-}\mathrm{ord}_{i-1} > 0$$

such that

$$\max\ w\text{-}\mathrm{ord}_0 \geq \max\ w\text{-}\mathrm{ord}_1 \geq \cdots \geq \max\ w\text{-}\mathrm{ord}_{k-1} \geq \max\ w\text{-}\mathrm{ord}_k$$

and that

$$\mathrm{Sing}(J_k, b) = \emptyset,$$

i.e., the sequence represents resolution of singularities of the basic object $(W_0, (J_0, b), E_0)$.

Proof.

A basic object $(W_0, (J_0, b), E_0)$ defines a general basic object $(\mathcal{F}_0, (W_0, E_0))$ with a d-dimensional structure, as explained in Remark 4-2 (i). A sequence representing resolution of singularities of $(\mathcal{F}_0, (W_0, E_0))$, obtained via the inductive algorithm of Theorem 5-1, provides that of $(W_0, (J_0, b), E_0)$ with the required properties.

CHAPTER 6. A MORE DOWN-TO-EARTH APPROACH TO THE INDUCTIVE ALGORITHM

In Chapter 5 we saw how the inductive algorithm for resolution of singularities of a (general) basic object works. However, in the untrained eyes (e.g. those of the author), its mechanism may look more like a miracle than a natural process. Especially the ingenious t-invariant seems to have come "out of blue". The purpose of this chapter is to explain the mathematical origin of the t-invariant and show how natural the inductive algorithm is.

For this purpose, firstly we break up the problem of resolution of singularities of a (general) basic object into the following three stages, depending upon the restrictions on the basic objects to deal with (and hence with increasing difficulties, going from A_d through B_d to C_d).

The descriptions of the restrictions we put at these stages on the basic objects are:

A_d: simple, with empty boundary, d-dimensional,
B_d: simple, with possibly non-empty boundary, d-dimensional, and
C_d: the general d-dimensional case with no restrictions, i.e., not necessarily simple, with possibly non-empty boundary, d-dimensional.

Secondly we establish the reduction steps.

Reduction A_d to C_{d-1}: via the key inductive lemma,
Reduction B_d to C_{d-1} (+ A_d): via the key inductive lemma and the introduction of the n-invariant,
Reduction C_d to B_d: via the introduction of the invariant w-ord,

and hence establish the inductive algorithm for resolution of singularities of a (general) basic object.

We denote the three reduction steps as above figuratively by

$\boxed{\text{Reduction } A_d \leftarrow C_{d-1}}$
$\boxed{\text{Reduction } B_d \leftarrow C_{d-1}}$
$\boxed{\text{Reduction } C_d \leftarrow B_d}$

where, e.g., $A_d \leftarrow C_{d-1}$ indicates that a solution to the problem of resolution of singularities at stage C_{d-1} implies a solution to the problem of resolution of singularities at stage A_d, with the arrow $\leftarrow$ representing the implication.

(We note and emphasize that logically we only need the last two reduction steps $C_d \leftarrow B_d, B_d \leftarrow C_{d-1}$ to complete the inductive algorithm and not the first $A_d \leftarrow C_{d-1}$, whose construction in showing the reduction step, but not whose consequence, is used in the argument to show the reduction step $B_d \leftarrow C_{d-1}$. The first reduction step is included solely to demonstrate how the key inductive lemma directly and easily solves the problem of resolution of singularities for a simple basic object with an empty boundary by induction.)

We observe that the presentation of the inductive algorithm in this chapter is a decomposition of the one in Chapter 5 into a few reduction steps as above and that the t-invariant is the natural combination of the n-invariant and the invariant

w-ord when one wants to put these reduction steps together back into one nice package as presented in Chapter 5.

This chapter is based upon one of the lectures delivered by Prof. Villamayor under the title “Constructive Desingularization” at Purdue University.

We start with giving a more precise description of the three stages A_d, B_d, and C_d of the problem of resolution of singularities of a (general) basic object.

Description of A_d, B_d, and C_d

$\underline{A_d : \text{ simple, with empty boundary, } d\text{-dimensional}}$

Resolution of singularities of a simple (general) basic object

$$(W,(J,b),E)$$

with an empty boundary divisor of dimension d, i.e.,

$$\begin{cases} \dim W = d, \\ \qquad E = \emptyset, \\ \nu_p(J) = b \quad \forall p \in \mathrm{Sing}(J,b). \end{cases}$$

(In case of a general basic object given by the charts $\{(\widetilde{W_0^\lambda},(\mathfrak{a}_0^\lambda,b^\lambda),\widetilde{E_0^\lambda})\}$ we require that all the charts are simple basic objects with empty boundary divisors of dimension d, i.e.,

$$\forall \lambda \in \Lambda \quad \begin{cases} \dim \widetilde{W_0^\lambda} = d, \\ \qquad \widetilde{E_0^\lambda} = \emptyset, \\ \nu_p(\mathfrak{a}_0^\lambda) = b^\lambda \quad \forall p \in \mathrm{Sing}(\mathfrak{a}_0^\lambda, b^\lambda).) \end{cases}$$

$\underline{B_d : \text{ simple, with possibly non-empty boundary, } d\text{-dimensional}}$

Resolution of singularities of a simple (general) basic object

$$(W,(J,b),E)$$

with a possibly non-empty boundary divisor of dimension d, i.e.,

$$\begin{cases} \dim W = d, \\ \qquad E \text{ an arbitrary divisor with simple normal crossings}, \\ \nu_p(J) = b \quad \forall p \in \mathrm{Sing}(J,b). \end{cases}$$

(In case of a general basic object given by the charts $\{(\widetilde{W_0^\lambda},(\mathfrak{a}_0^\lambda,b^\lambda),\widetilde{E_0^\lambda})\}$ we require that all the charts are simple basic objects with possibly non-empty boundary divisors of dimension d, i.e.,

$$\forall \lambda \in \Lambda \quad \begin{cases} \dim \widetilde{W_0^\lambda} = d, \\ \qquad \widetilde{E_0^\lambda} = E \cap \widetilde{W_0^\lambda}, \\ \qquad\quad \text{where } E \text{ is an arbitrary divisor with simple normal crossings,} \\ \qquad\quad \text{intersecting } \widetilde{W_0^\lambda} \text{ transversally,} \\ \nu_p(\mathfrak{a}_0^\lambda) = b^\lambda \quad \forall p \in \mathrm{Sing}(\mathfrak{a}_0^\lambda, b^\lambda).) \end{cases}$$

$\underline{C_d : \text{the general } d\text{-dimensional case with no restrictions, i.e.,}}$
$\underline{\text{not necessarily simple, with possibly non-empty boundary, } d\text{-dimensional}}$

Resolution of singularities of a (general) basic object

$$(W, (J, b), E)$$

without any restrictions of dimension d, i.e., it may not be simple and with a possibly non-empty boundary divisor.

(In case of a general basic object given by the charts $\{(\widetilde{W_0^\lambda}, (\mathfrak{a}_0^\lambda, b^\lambda), \widetilde{E_0^\lambda})\}$ there is no restriction on the basic objects $(\widetilde{W_0^\lambda}, (\mathfrak{a}_0^\lambda, b^\lambda), \widetilde{E_0^\lambda})$ other than $\dim \widetilde{W_0^\lambda} = d$, i.e., they may not be simple but with possibly non-empty boundary divisors, and of dimension d.)

Arguments for the reduction steps

Now we are ready to provide the arguments for the following three reduction steps:

$\boxed{\text{Reduction } A_d \leftarrow C_{d-1}}$
$\boxed{\text{Reduction } B_d \leftarrow C_{d-1}}$
$\boxed{\text{Reduction } C_d \leftarrow B_d}$.

$\boxed{\text{Reduction } A_d \leftarrow C_{d-1}}$

Let $(W, (J, b), E)$ be a simple basic object with $E = \emptyset$.

Then we can find an open covering $\{W^\lambda\}_{\lambda \in \Lambda}$ of W with smooth hypersurfaces $W_h^\lambda \subset W^\lambda$, embedded as closed subschemes, satisfying conditions 1 and 2 of the key inductive lemma. In fact, by Remark 1-5 with

$$S = \{p \in W; \nu_p(J) = b_{\max} = b\} = \mathrm{Sing}(J, b)$$

where $b_{\max} = b$ since $(W, (J, b), E)$ is simple, we can find an open covering $\{W^\lambda\}_{\lambda \in \Lambda}$ of W with smooth hypersurfaces $W_h^\lambda \subset W^\lambda$ satisfying condition 1. Now condition 2 is void and hence automatically satisfied, since E is empty.

Case A: $R(1)(\mathrm{Sing}(J, b)) \neq \emptyset$.

As in the proof of the key inductive lemma, it is easy to see in this case that $R(1)(\mathrm{Sing}(J, b))$ is smooth and open in $\mathrm{Sing}(J, b)$ and that it is permissible with respect to $E = \emptyset$ automatically. After taking the transformation with center $R(1)(\mathrm{Sing}(J, b))$, we are in **Case B**.

Case B: $R(1)(\mathrm{Sing}(J, b)) = \emptyset$.

Having an open covering $\{W^\lambda\}_{\lambda \in \Lambda}$ of W with smooth hypersurfaces $W_h^\lambda \subset W^\lambda$ satisfying conditions 1 and 2, we are in a position to apply the key inductive lemma to conclude that resolution of singularities of $(W, (J, b), E)$ is reduced to resolution of singularities of the general basic objects whose charts are given by $\{(\widetilde{W_0^\lambda}, (\mathfrak{a}_0^\lambda, b^\lambda), \widetilde{E_0^\lambda})\}$ as constructed in the key inductive lemma.

Thus we see that A_d is reduced to C_{d-1}.

Crucial Remark in A_d

Remark that if we have a sequence of transformations of basic objects

$$(W,(J,b),E=\emptyset)=(W_0,(J,b),E_0)\leftarrow\cdots\leftarrow(W_k,(J_k,b),E_k)$$

and if we have an open covering $\{W^\lambda\}_{\lambda\in\Lambda}$ with smooth hypersurfaces $W_h^\lambda\subset W^\lambda$ satisfying condition 1 (and 2), then the open covering $\{W_k^\lambda\}_{\lambda\in\Lambda}$ consisting of the inverse images W_k^λ of $W^\lambda=W_0^\lambda$ and the smooth hypersurfaces consisting of the strict transforms $(W_k^\lambda)_h\subset W_k^\lambda$ of $(W_0^\lambda)_h=W_h^\lambda\subset W^\lambda=W_0^\lambda$, satisfy conditions 1 and 2 of the key inductive lemma for the simple basic object $(W_k,(J_k,b),E_k)$. This is a consequence of Giraud's Lemma (cf. Claim 3-4), which is an essential part of the construction in the key inductive lemma.

$\boxed{\text{Reduction } B_d\leftarrow C_{d-1}}$

Let $(W,(J,b),E)$ be a simple basic object with $E=\{H_1,...,H_r\}$.

Let

$$(W,(J,b),E)=(W_0,(J_0,b),E_0)\leftarrow(W_1,(J_1,b),E_1)\leftarrow\cdots\leftarrow(W_k,(J_k,b),E_k)$$

be a sequence of transformations of basic objects. We decompose $E_k=E_k^-\cup E_k^+$ into two disjoint subsets $E_k^-=\{H_1,...,H_r\}$ and its complement $E_k^+=\{H_{r+1},...,H_{r+k}\}$ in $E_k=\{H_1,...,H_r,H_{r+1},...,H_{r+k}\}$. (Look at the convention in Definition 1-8 (iii) and see also Definition 1-10 (iii).) Note that the assumption $(W,(J,b),E)$ being simple implies $(W_i,(J_i,b),E_i)$ also being simple and $w\text{-ord}_i=\text{ord}_i$ for $i=0,...,k$.) Remark that this decomposition is motivated by the crucial remark at the end of the discussion of A_d.

Define the function (cf. Definition 1-10 (iii))

$$n_k:\text{Sing}(J_k,b)\to\mathbb{Z}_{\geq 0}$$

by

$$n_k(p)=\#\{H_i\in E_k^-;p\in H_i\}\text{ for }p\in\text{Sing}(J_k,b).$$

Case: max $n_k=0$.

Observe that corresponding to the original sequence of transformations of basic objects

$$(W,(J,b),E)=(W_0,(J_0,b),E_0)\leftarrow(W_1,(J_1,b),E_1)\leftarrow\cdots\leftarrow(W_k,(J_k,b),E_k)$$

we have another sequence of transformations of basic objects

$$\begin{aligned}(W,(J,b),\emptyset)=(W_0,(J_0,b),E_0^+)\leftarrow(W_1,(J_1,b),(E_0^+)_1=E_1^+)\leftarrow\cdots\\ \cdots\leftarrow(W_k,(J_k,b),(E_0^+)_k=E_k^+).\end{aligned}$$

Since $\mathrm{Sing}(J_k, b) \cap E_k^- = \emptyset$ under the case assumption max $n_k = 0$, we conclude that a sequence representing resolution of singularities of $(W_k, (J_k, b), E_k^+)$ is also a sequence representing resolution of singularities of $(W_k, (J_k, b), E_k)$. Thus we only have to find a sequence representing resolution of singularities of $(W_k, (J_k, b), E_k^+)$.

We now observe, by the crucial remark at the end of the reduction step $A_d \leftarrow C_{d-1}$, that we can find an open covering $\{W_k^\lambda\}_{\lambda \in \Lambda}$ of W_k with smooth hypersurfaces $(W_k^\lambda)_h \subset W_k^\lambda$, embedded as closed subschemes, satisfying conditions 1 and 2 of the key inductive lemma.

Case A: $R(1)(\mathrm{Sing}(J_k, b)) \neq \emptyset$.

Again as in the proof of the key inductive lemma, it is easy to see in this case that $R(1)(\mathrm{Sing}(J, b))$ is smooth and open in $\mathrm{Sing}(J_k, b)$ and that it is permissible with respect to E_k^+, as so is $(W_k^\lambda)_h$. After taking the transformation with center $R(1)(\mathrm{Sing}(J_k, b))$, we are in **Case B**.

Case B: $R(1)(\mathrm{Sing}(J_k, b)) = \emptyset$.

Having an open covering $\{W_k^\lambda\}_{\lambda \in \Lambda}$ of W_k with smooth hypersurfaces $(W_k^\lambda)_h \subset W_k^\lambda$ satisfying conditions 1 and 2, we are in a position to apply the key inductive lemma to conclude that resolution of singularities of $(W_k, (J_k, b), E_k^+)$ is reduced to resolution of singularities of the general basic objects whose charts are given by $\{(\widetilde{W_k^\lambda}, (\mathfrak{a}_k^\lambda, b^\lambda), \widetilde{(E_k^+)^\lambda})\}$ as constructed in the key inductive lemma.

Thus we see that B_d is reduced to C_{d-1} via the key inductive lemma in this case of max $n_k = 0$.

Case: max $n_k = l_k > 0$.

Consider the following basic object $(V, (I, c), G)$
where $(V, (I, c), G) = (W_k{}'', (J_k{}'', b''), E_k{}'')$ in the notation of Lemma 5-4, i.e.,

$$\begin{cases} V = W_k \\ I = (J_k) + \displaystyle\prod_{\{H_{s_1}, \ldots, H_{s_{l_k}}\} \subset E_k^- \text{ with } H_{s_1}, \ldots, H_{s_{l_k}} \text{ distinct}} \{I(H_{s_1}) + \cdots I(H_{s_{l_k}})\}^b \\ c = b \\ G = E_k^+. \end{cases}$$

(Remark that, since $(W, (J, b), E) = (W_0, (J_0, b), E_0)$ is simple, so is $(W_i, (J_i, b), E_i)$ and $\mathrm{ord}_i = w\text{-}\mathrm{ord}_i$ & $J_i = \overline{J_i}$ for $i = 0, ..., k$. This implies
$b_k = b \cdot \max\ w\text{-}\mathrm{ord}_k = b \cdot \mathrm{ord}_k = b = b' = b''$ in the notation of Lemma 5-3 and Lemma 5-4).

First it is easy to see that $(V, (I, c), G)$ is a simple basic object (as shown in Lemma 5-4). Second we claim that for the simple basic object $(V, (I, c), G)$ we can find an open covering $\{V^\lambda\}_{\lambda \in \Lambda}$ of V with smooth hypersurfaces $V_h^\lambda \subset V^\lambda$ satisfying conditions 1 and 2 of the key inductive lemma. In fact, by the crucial reamrk at the end of the reduction step $A_d \leftarrow C_{d-1}$, for the simple basic object $(W_k, (J_k, b), E_k^+)$ we can find an open covering $\{W_k^\lambda\}_{\lambda \in \Lambda}$ of W_k with smooth hypersurfaces $(W_k^\lambda)_h \subset W_k^\lambda$ satisfying conditions 1 and 2 of the key inductive lemma. Set

$$V^\lambda = W_k^\lambda, V_h^\lambda = (W_k^\lambda)_h.$$

Then we have for the simple basic object $(V, (I, c), G)$

$$I(V_h^\lambda) = I((W_k^\lambda)_h) \subset \Delta^{b-1}(J_k) \subset \Delta^{c-1}(I),$$

and hence satisfying condition 1. Condition 2 is identical and satisfied both for the simple basic object $(V, (I, c), G)$ and for $((W_k, (J_k, b), E_k^+)$, since $G = E_k^+$.

Therefore, by the key inductive lemma and C_{d-1}, possibly after going through **Case A** first, we find a sequence representing resolution of singularities of $(V, (I, c), G)$

$$(V, (I, c), G) = (V_0, (I_0, c), G_0) \leftarrow \cdots \leftarrow (V_N, (I_N, c), G_N)$$

where $\mathrm{Sing}(J_N, c) = \emptyset$. Then we observe that there corresponds an extension of the original sequence of transformations with the same centers

$$(W_k, (J_k, b), E_k) \leftarrow \cdots \leftarrow (W_{k+N}, (J_{k+N}, b), E_{k+N}),$$

where inductively for $j = 0, ..., N$ we see that

$$I_j = (J_{k+j}) + \prod_{\{H_{s_1}, ..., H_{s_{l_k}}\} \subset E_{k+j}^- \text{ with } H_{s_1}, ..., H_{s_{l_k}} \text{ distinct}} \{I(H_{s_1}) + \cdots I(H_{s_{l_k}})\}^b$$

and that for $j = 1, ..., N$ the center $Y_{k+j-1} = Y_{G,j-1}$ is permissible with respect to $(W_{k+j-1}, (J_{k+j-1}, b), E_k)$, since

$$Y_{G,j-1} \subset \mathrm{Sing}(I_{j-1}, c) \subset \mathrm{Sing}(J_{k+j-1}, b),$$

since $Y_{G,j-1}$ is contained in $H_s \in E_{k+j-1}^-$ if $Y_{G,j-1} \cap H_s \neq \emptyset$ as $\max\ n_k = l_k = l_{k+j-1} = \max\ n_{k+j-1}$, and since $Y_{G,j-1}$ is permissible with respect to $G_{j-1} = E_{k+j-1}^+$.

Finally, since

$$I_N = (J_{k+N}) + \prod_{\{H_{s_1}, ..., H_{s_{l_k}}\} \subset E_{k+N}^- \text{ with } H_{s_1}, ..., H_{s_{l_k}} \text{ distinct}} \{I(H_{s_1}) + \cdots I(H_{s_{l_k}})\}^b$$

and since

$$\mathrm{Sing}(I_N, c) = \emptyset,$$

we conclude that either

$$\mathrm{Sing}(J_{k+N}, b) = \emptyset,$$

in which case the extension realizes a sequence representing resolution of singularities of $(W, (J, b), E)$, or

$$\max\ n_k > \max\ n_{k+N},$$

in which case by induction on the maximum of the invariant n we also obtain a sequence representing resolution of singularities of $(W, (J, b), E)$.

This completes the proof of the reduction step $B_d \leftarrow C_{d-1}$ via the key inductive lemma and the introduction of the invariant n.

$\boxed{\text{Reduction } C_d \leftarrow B_d}$

Let $(W, (J, b), E)$ be a basic object without any restrictions except that $\dim W = d$, i.e., it may not be simple but with a possibly non-empty boundary divisor, and of dimension d.

Let

$$(W, (J, b), E) = (W_0, (J_0, b), E_0) \leftarrow (W_1, (J_1, b), E_1) \leftarrow \cdots \leftarrow (W_k, (J_k, b), E_k)$$

be a sequence of transformations of basic objects.

Define the function

$$w\text{-ord}_k : \mathrm{Sing}(J_k, b) \to \frac{1}{b}\mathbb{Z}_{\geq 0}$$

by

$$w\text{-ord}_k(p) = \frac{\nu_p(\overline{J_k})}{b} \text{ for } p \in \mathrm{Sing}(J_k, b)$$

as in Definition 1-10 (ii).

Case: max w-ord$_k = 0$.

In this case, the problem of resolution of singularities of $(W_k, (J_k, b), E_k)$ is reduced to that of monomial basic objects, which is settled in Chapter 2.

Case: max w-ord$_k > 0$.

In this case, consider the following basic object $(V, (I, c), G)$ where $(V, (I, c), G) = (W'_k, (J'_k, b'), E'_k)$ in the notation of Lemma 5-3, i.e.,

$$\begin{cases} V = W'_k = W_k \\ I = J'_k = \begin{cases} \overline{J_k} & \text{if } b_k \geq b \\ \overline{J_k}^{b-b_k} + \{I(H_{r+1})^{a_{r+1}} \cdots I(H_{r+k})^{a_{r+k}}\}^{b_k} & \text{if } b_k < b \end{cases} \\ c = b' = \begin{cases} b_k & \text{if } b_k \geq b \\ b_k(b - b_k) & \text{if } b_k < b \end{cases} \\ G = E'_k = E_k. \end{cases}$$

Recall that

$$\begin{aligned} J_k &= I(H_{r+1})^{a_{r+1}} \cdots I(H_{r+k})^{a_{r+k}} \cdot \overline{J_k} \\ b_k &= b \cdot (\max\ w\text{-ord}_k). \end{aligned}$$

Then $(V, (I, c), G)$ is a simple basic object of dimension d (as shown in Lemma 5-3).

Therefore, by B_d, we find a sequence representing resolution of singularities of $(V, (I, c), G)$

$$(V, (I, c), G) = (V_0, (I_0, c), G_0) \leftarrow \cdots \leftarrow (V_N, (I_N, c), G_N)$$

where $\mathrm{Sing}(J_N, c) = \emptyset$. Then we observe that there corresponds an extension of the original sequence of transformations with the same centers

$$(W_k, (J_k, b), E_k) \leftarrow \cdots \leftarrow (W_{k+N}, (J_{k+N}, b), E_{k+N}),$$

where inductively for $j = 0, ..., N-1$ we see that

$$b_k = b_{k+j}, \text{ i.e. }, \max\ w\text{-ord}_k = \max\ w\text{-ord}_{k+j} \quad \& \quad I_j = J'_{k+j}$$

and for $j = N$

$$I_N = \begin{cases} \overline{J_{k+N}} & \text{if } b_{k+N-1} = b_k \geq b \\ \overline{J_{k+N}}^{b-b_{k+N-1}} + \{I(H_{r+1})^{a_{r+1}} \cdots I(H_{r+k+N})^{a_{r+k+N}}\}^{b_{k+N-1}} & \text{if } b_{k+N-1} = b_k < b, \end{cases}$$

and that for $j = 1, ..., N$ the center

$$Y_{k+j-1} = Y_{G,j-1} \subset \mathrm{Sing}(I_{j-1}, c) = \mathrm{Sing}(J'_{k+j-1}, c) \subset \mathrm{Sing}(J_{k+j-1}, b)$$

is permissible with respect to $E_{k+j-1} = G_{j-1}$.

Finally since

$$\mathrm{Sing}(I_N, c) = \emptyset,$$

we conclude that either

$$\mathrm{Sing}(J_{k+N}, b) = \emptyset,$$

in which case the extension realizes a sequence representing resolution of singularities of $(W, (J, b), E)$, or

$$\max\ w\text{-ord}_k > \max\ w\text{-ord}_{k+N},$$

in which case by induction on the maximum of the invariant w-ord we also obtain a sequence representing resolution of singularities of $(W, (J, b), E)$.

This completes the proof of the reduction step $C_d \leftarrow B_d$.

(The argument for resolution of singualrities of a general basic object is identical and left to the reader as an exercise.)

Remark 6-1.

Though suppressed in the above argument of the reduction steps, it is absolutely necessary and crucial to argue and verify that the processes of resolution of singularities of charts in the general basic object patch up, via the observation that our choice of the centers only depend on the invariants we set up and that those invariants are independent of charts, which is one of the essential points in Chapter 4 via Hironaka's trick. It is also necessary and crucial to generalize the notion of a basic object to that of a general basic object to complete the inductive step.

Exercise 6-2.

Check that the inductive algorithm for resolution of singularities given by the reduction steps $C_d \leftarrow B_d$, $B_d \leftarrow C_{d-1}$ described as above actually coincides with the one described in Chapter 5 using the t-invariant.

CHAPTER 7. EMBEDDED RESOLUTION OF SINGULARITIES

In this chapter, we present a proof for (embedded resoluion of singularities) stated in Main Theme 0-2, as an easy consequence of (resolution of singularities of a basic object) proved in Corollary 5-6.

Theorem 7-1 (Embedded resolution of singularities). *Let $X \subset W$ be a variety, embedded as a closed subscheme of another variety W smooth over a field k of charactersitic zero.*

We can construct a sequence of blowups

$$X = X_0 \subset W = W_0 \overset{\pi_1}{\leftarrow} X_1 \subset W_1 \overset{\pi_2}{\leftarrow} \cdots \overset{\pi_{l-1}}{\leftarrow} X_{l-1} \subset W_{l-1} \overset{\pi_l}{\leftarrow} X_l \subset W_l$$

so that

(i) the centers $Y_{i-1} \subset W_{i-1}$ of the blowups $\pi_i \quad (i = 1, ..., l)$ are over $\mathrm{Sing}(X) = X \setminus \mathrm{Reg}(X)$,

(ii) the centers $Y_{i-1} \subset W_{i-1}$ are permissible with respect to the exceptional divisors $E_{i-1} \subset W_{i-1}$ for the morphisms $\psi_{i-1} = \pi_1 \circ \pi_2 \circ \cdots \circ \pi_{i-2} \circ \pi_{i-1}$ (which are simple normal crossing divisors),

(iii) the strict transform X_l (of X_0) $\subset W_l$ is a variety smooth over k, permissibe with respect to E_l, and the induced morphism $X = X_0 \overset{\pi}{\leftarrow} X_l$, where $\pi = \psi_l = \pi_1 \circ \pi_2 \circ \cdots \circ \pi_{l-1} \circ \pi_l$, is a projective birational morphism isomorphic over $\mathrm{Reg}(X)$.

Remark 7-2.

(i) We want to emphasize that we do NOT require that $Y_{i-1} \subset X_{i-1}$, i.e., the center Y_{i-1} be contained in the strict transform X_{i-1} of $X = X_0$, or that it be contained in its singular locus, i.e., $Y_{i-1} \subset \mathrm{Sing}(X_{i-1})$, as Hironaka or Bierstone-Milman does in their formulation of embedded resolution of singularities. Therefore, though the centers Y_{i-1} are smooth in the ambient varieties W_{i-1} and permissible with respect to E_{i-1}, their restrictions $Y_{i-1} \cap X_{i-1}$ to the strict transforms may not be smooth or reduced in general. See Chapter 11 for some examples.

(ii) When X is a hypersurface in W, i.e., $\dim X = \dim W - 1$, our algorithm provides a sequence satisfying

(i') the centers $Y_{i-1} \subset W_{i-1}$ of the blowups $\pi_i \quad (i = 1, ..., l)$ are contained in $\mathrm{Sing}(X_{i-1})$,

which is stronger than condition (i) and coincides with the requirement that Hironaka or Bierstone-Milman makes.

This is because, in the case of X being a hypersurface, the weak transform coincides with the strict transform and because our centers are contained in the maximum loci of the invariant w-ord (by construction), which are necessarily sitting inside of the singular loci of $X_{i-1} \quad (i = 1, ..., l)$.

Proof.

We consider the following basic object $(W_0,(J_0,b),E_0)$ where

$$\begin{cases} W_0 = W \\ J_0 = \mathcal{I}_X \quad \text{(the defining ideal of } X \text{ in } W) \\ b = 1 \\ E_0 = \emptyset \end{cases}$$

and take the sequence of transformations of basic objects, which represents resolution of singularities of $(W_0,(J_0,b),E_0)$, constructed by the inductive algorithm of Theorem 5-1 and with the properties as specified in Corollary 5-6

$$(W_0,(J_0,b),E_0) \leftarrow \cdots \leftarrow (W_k,(J_k,b),E_k).$$

Observe that if for $i=1,...,l$ with $l \leq k$ the centers $Y_{i-1} \subset W_{i-1}$ do not contain the strict transforms X_{i-1}, then X_l is an irreducible component of $\mathrm{Sing}(J_l,b)$. Since $\mathrm{Sing}(J_k,b)=\emptyset$, we conclude that there exists $1 \leq l \leq k-1$ such that Y_l contains X_l, while Y_{i-1} does not contain X_{i-1} for $i=1,...,l$. Moreover, since $Y_l \subset \mathrm{Sing}(J_l,b)$, we see that X_l is an irreducible component of Y_l. Since Y_l is smooth over k, a property which is guaranteed by the inductive algorithm, we conclude that X_l is smooth over k.

If we look at the sequence up to the l-th stage, it is immediate that it satisfies conditions (ii) and (iii) (except for the claim that π is isomorphic over $\mathrm{Reg}(X)$, which follows once we check condition (i)).

Condition (i) is a consequence of the process prescribed by the inductive algorithm of Theorem 5-1. In fact, let $p \in \mathrm{Reg}(X)$ be an arbitrary point of $\mathrm{Reg}(X)$. Let l_p be the smallest number so that $p \in Y_{l_p}$. The condition $(\heartsuit')$

$$(\heartsuit') \quad Y_{i-1} \subset \underline{\mathrm{Max}}\ t_{i-1} \subset \underline{\mathrm{Max}}\ w\text{-}\mathrm{ord}_{i-1} \text{ if } w\text{-}\mathrm{ord}_{i-1} > 0 \text{ for } i=1,...,k$$

implies that

$$\max\ w\text{-}\mathrm{ord}_{l_p} = 1 = w\text{-}\mathrm{ord}_{l_p}(p) \quad \& \quad \max\ t_{l_p} = t_{l_p}(p) = (1,0).$$

There exists an open neighborhood $p \in U_p = W^{\lambda}_{l_p} \subset W_{l_p}$ such that $\mathrm{Reg}(X_{l_p}) \cap U_p = V(x_1,\cdots,x_r)$ where $x_1,\cdots,x_r$ are regular parameters with $r = \dim W_{l_p} - \dim X_{l_p}$. According to the inductive algorithm described in Theorem 5-1, after $(r-1)$-repetitions of **Case B**, we reach a $(\dim X_{l_p}+1)$-dimensional basic object $(\widetilde{W^{\lambda}_{l_p}},(\mathfrak{a}^{\lambda}_{l_p},b^{\lambda}),\widetilde{E^{\lambda}_{l_p}}) = (V(x_1,\cdots,x_{r-1}),((x_r),1),\emptyset)$ where $R(1)(\underline{\mathrm{Max}}\ t_{l_p}) \cap W^{\lambda}_{l_p} = V(x_1,\cdots,x_r) = \mathrm{Reg}(X_{l_p}) \cap U_p$ and hence we are in **Case A**. This implies that X_{l_p} is contained in the center Y_{l_P} and hence that $l_p = l$.

Since l_p is the smallest number so that $p \in Y_{l_p}$ and since $p \in \mathrm{Reg}(X)$ is arbitrary, we conclude that the centers $Y_{i-1} \subset W_{i-1}$ of the blowups $\pi_i \quad (i=1,...,l)$ are over $\mathrm{Sing}(X) = X \setminus \mathrm{Reg}(X)$, verifying condition (i).

This completes the proof of Theorem 7-1.

Remark 7-3.

(i) Resolution of singularities of a basic object $(W, (\mathcal{I}_X, 1), \emptyset)$ is called the "**principalization**" of the ideal $\mathcal{I}_X$, since as a consequence we obtain

$$\pi^{-1}\mathcal{I}_X \cdot \mathcal{O}_{W_k} = I(H_{r+1})^{a_{r+1}} \cdots I(H_{r+k})^{a_{r+k}}$$

making the total transform of the ideal "principal" (where actually $r = 0$ as $E_0 = \{H_1, ..., H_r\} = \emptyset$). (See the remark right after Main Theme 0-3 (Principalization of ideals for our restrictive use of the word "principal".)

(ii) In the paper "A course on constructive desingularization and equivariance", embedded resolution of *hypersurface* singularities is proved, starting with resolution of singularities of a basic object $(W_0, (J_0, b), E_0)$ where

$$\begin{cases} W_0 = W \\ J_0 = \mathcal{I}_X \\ b = b_{\max} = \max\{\nu_p(\mathcal{I}_X); p \in W_0\} \\ E_0 = \emptyset \end{cases}$$

and then continuing with the descending induction on $b_{\max}$. Embedded resolution of *non-hypersurface* singularities quotes the results of Hironaka and others (without proof) which use the Hilbert-Samuel function. It was a big discouragement for those of us in the seminar who were hoping to have a self-contained course with complete proofs for the entire picture of the process of resolution of (*hypersurface* and *non-hypersurface*) singularities.

But then after moments of thoughts, we the students realize, as the teachers Encinas and Villamayor reveal to us[1] in the second paper "A new theorem of desingularization over fields of characteristic zero", that they have already told us ALL in the first paper, i.e., the inductive algorithm of resolution of singularities of general basic objects applied to the principalization of the ideal $\mathcal{I}_X$ gives us reslotuion of singularities, *hypersurface* and *non-hypersurface* , even **without** the use of Hilbert-Samuel function. (Of course, however, a price has to be paid in not being able to claim the stronger properties on the centers in the process of resolution of singularities, as mentioned in Remark 7-2.)

[1]Prof. Bierstone informed us in an informal way that this idea of achieving resolution of singularities, *hypersurface* and *non-hypersurafce* , as a consequence of principalization imposing the permissibility condition on the centers, could also be traced back to Hironaka.

CHAPTER 8. EQUIVARIANCE AND RESOLUTION OF SINGULARITIES OVER BASE FIELDS (OF CHARACTERISTIC ZERO) WHICH ARE POSSIBLY NOT ALGEBRAICALLY CLOSED

In this chapter, we prove that the inductive algorithm for resolution of singularities of (general) basic objects presented in Chapter 5 is equivariant under any "action" (of a group) and hence that all the centers are invariant under the action. This implies, as an easy corollary, that given a basic object $(W,(J,b),E)$ defined over a field k which is of characteristic zero but which may not be algebraically closed, the inductive algorithm of resolution of singularities for the basic object $(W,(J,b),E) \times \operatorname{Spec} \overline{k}$ is equivariant under the action of the Galois group $\operatorname{Gal}(\overline{k}/k)$, all the centers are invariant under the action of $\operatorname{Gal}(\overline{k}/k)$ and hence defined over k and that it induces resolution of singularities of $(W,(J,b),E)$ over k.

Definition 8-1 ("Action" on a basic object). *Let $(W, E = \{H_1, ..., H_r\})$ and $(W', E' = \{H'_1, ..., H'_{r'}\})$ be pairs (cf. Definition 1-6). An isomorphism of pairs $\theta : (W,E) \xrightarrow{\sim} (W',E')$ is an isomorphism $\theta : W \xrightarrow{\sim} W'$ as abstract varieties (not necessarily over the base field k) such that $r = r'$ and that*

$$\theta(H_i) = H'_i \text{ for } i = 1, ..., r.$$

Let $(W,(J,b),E)$ and $(W',(J',b'),E')$ be basic objects. An isomorphism of basic objects $\theta : (W,(J,b),E) \xrightarrow{\sim} (W',(J',b'),E')$ is an isomorphism of pairs $\theta : (W,E) \xrightarrow{\sim} (W',E')$ such that $b' = b$ and that it induces an isomorphism of ideals

$$J' = \theta_*(J) \subset \theta_*(\mathcal{O}_W) = \mathcal{O}_{W'}.$$

An action on a pair (W,E), by definition, is an isomorphism of pairs of (W,E) onto itself. An action on a basic object $(W,(J,b),E)$, by definition, is an isomorphism of basic objects of $(W,(J,b),E)$ onto itself.

Remark 8-2.

(i) Recall that H_i actually consists of smooth irreducible components $H_{i,1}, ..., H_{i,l_i}$ in our notation (cf. Note 1-7) and that so does H'_i of $H'_{i,1}, ..., H'_{i,l_i'}$. Therefore, when we state the condition

$$\theta(H_i) = H'_i \text{ for } i = 1, ..., r,$$

what we really mean is that for each $i = 1, \cdots, r$ we have $l_i = l'_i$ and that there is a permutation of $\{1, \cdots, l_i\}$ (which we denote by the same letter θ by abuse of notation) with

$$\theta(H_{i,j}) = H'_{i,\theta(j)} \text{ for } j = 1, ..., l_i.$$

(ii) As stated above, in order for θ to be an action on a pair, we do require not only E to be preserved as a whole but also each H_i to be preserved by θ, fixing each index i. Since our algorithm for resolution of singularities of (monomial and hence all the general) basic objects depends on the indexing of the divisors in E (cf. Definition 2-3, Proposition 2-5, Corollary 2-6, Definition-Proposition 4-5 and

Theorem 5-1), this requirement is necessary for us to claim that our algorithm for resolution of singularities of (general) basic objects is equivariant under any action. However, this requirement makes little difference when we consider the equivariance of (embedded or non-embedded) resolution of singulaities of a variety X, since the basic object of concern $(W, (\mathcal{I}_X, 1), \emptyset)$ that we start with (cf. Chapter 7) has empty boundary divisor $E = \emptyset$ and since the indexing of the subsequent exceptional divisors are determined by the resolution process itself.

(iii) We do NOT require in the definition of an action for an isomorphism θ to be over the base field k.

(iv) Two non-isomorphic basic objects may define isomorphic general basic objects: Take $(W, (J, b), E) = (\mathbb{A}^2 = \text{Spec } k[x, y], ((x), 1), \emptyset)$ and $(W', (J', b'), E') = (\mathbb{A}^2, ((x^2), 2), \emptyset)$. Then $(W, (J, b), E)$ and $(W', (J', b'), E')$ are non-isomorphic as basic objects, though they define the same general basic object representing the same collection $\mathfrak{C}$ of sequences of transformations and smooth morphisms of pairs with specified closed subsets according to Remark 4-2 (ii) (cf. Definition 8-5).

Definition 8-3 (Equivariant sequence of basic objects). *Let*

$$\begin{gathered}(W, (J, b), E) = (W_0, (J_0, b), E_0) \stackrel{\pi_1}{\leftarrow} (W_1, (J_1, b), E_1) \stackrel{\pi_2}{\leftarrow} \cdots \\ (W_{i-1}, (J_{i-1}, b), E_{i-1}) \stackrel{\pi_i}{\leftarrow} (W_i, (J_i, b), E_i) \\ \cdots \stackrel{\pi_{k-1}}{\leftarrow} (W_{k-1}, (J_{k-1}, b), E_{k-1}) \stackrel{\pi_k}{\leftarrow} (W_k, (J_k, b), E_k)\end{gathered}$$

be a sequence of transformations and smooth morphisms of basic objects.

We say that the sequence is θ-equivariant, given an action θ on the basic object $(W, (J, b), E)$, if inductively for $i = 1, ..., k$ we have a commutative diagram

$$\begin{array}{ccc} (W_{i-1}, (J_{i-1}, b), E_{i-1}) & \stackrel{\pi_i}{\leftarrow} & (W_i, (J_i, b), E_i) \\ {\scriptstyle\theta}\downarrow & & {\scriptstyle\theta}\downarrow \\ (W_{i-1}, (J_{i-1}, b), E_{i-1}) & \stackrel{\pi_i}{\leftarrow} & (W_i, (J_i, b), E_i) \end{array}$$

i.e., the action $\theta : (W_{i-1}, (J_{i-1}, b), E_{i-1}) \stackrel{\sim}{\rightarrow} (W_{i-1}, (J_{i-1}, b), E_{i-1})$ induced by $\theta : (W_0, (J_0, b), E_0) \stackrel{\sim}{\rightarrow} (W_0, (J_0, b), E_0)$ lifts to an action $\theta : (W_i, (J_i, b), E_i) \stackrel{\sim}{\rightarrow} (W_i, (J_i, b), E_i)$.

We note that if the sequence is θ-equivariant, then

$$\begin{cases} \theta(Y_{i-1}) = Y_{i-1} \\ \textit{whenever } \pi_i \textit{ is a transformation with center } Y_{i-1} \subset \text{Sing}(J_{i-1}, b) \subset W_{i-1}. \end{cases}$$

We say that the sequence is equivariant if it is θ-equivariant for any *action θ on $(W, (J, b), E)$.*

Definition 8-4 (Equivariant resolution of singularities of a basic object). *Let*

$$(W, (J, b), E) = (W_0, (J_0, b), E_0) \leftarrow \cdots \leftarrow (W_k, (J_k, b), E_k)$$

be a sequence of transformations representing resolution of singularities of the basic object $(W, (J, b), E)$ (i.e., $\text{Sing}(J_k, b) = \emptyset$).

We say that the resolution of singularities is θ-equivariant, given an action θ on $(W, (J, b), E)$, if the sequence is θ-equivariant.

We say that the resolution of singularities is equivariant if the sequence is equivariant.

Definition 8-5 ("Action" on a general basic object). *Let $(\mathcal{F}_0, (W_0, E_0))$ be a general basic object over $(F_0, (W_0, E_0))$ with a d-dimensional structure, representing the collection $\mathfrak{C}$ of sequences of transformations and smooth morphisms of pairs with specified closed subsets. Let $(\mathcal{F}'_0, (W'_0, E'_0))$ be another over $(F'_0, (W'_0, E'_0))$ with a $d' = d$-dimensional structure, representing the collection $\mathfrak{C}'$. An isomorphism of general basic objects $\theta : (\mathcal{F}_0, (W_0, E_0)) \xrightarrow{\sim} (\mathcal{F}'_0, (W'_0, E'_0))$ is an isomorphism of pairs $\theta : (W_0, E_0) \xrightarrow{\sim} (W'_0, E'_0)$ with $\theta(F_0) = F'_0$ which satisfies the following condition:*

For each commutative diagram of sequences of transformations and smooth morphisms of pairs with specified closed subsets

$$\begin{array}{ccc} (F_0, (W_0, E_0)) \leftarrow & \cdots & \leftarrow (F_k, (W_k, E_k)) \\ \theta \downarrow & & \theta \downarrow \\ (F'_0, (W'_0, E'_0)) \leftarrow & \cdots & \leftarrow (F'_k, (W'_k, E'_k)), \end{array}$$

where the vertical arrows are all isomorphisms of pairs $\theta : (W_i, E_i) \xrightarrow{\sim} (W'_i, E'_i)$ with $\theta(F_i) = F'_i$, induced by the original isomorphism of pairs $\theta : (W_0, E_0) \xrightarrow{\sim} (W'_0, E'_0)$ with $\theta(F_0) = F'_0$, the sequence

$$(F_0, (W_0, E_0)) \leftarrow \cdots \leftarrow (F_k, (W_k, E_k))$$

is in the collection $\mathfrak{C}$ if and only if the sequence

$$(F'_0, (W'_0, E'_0)) \leftarrow \cdots \leftarrow (F'_k, (W'_k, E'_k))$$

is in the collection $\mathfrak{C}'$.

(By abuse of notation, we could express the last condition as requiring $\theta(\mathfrak{C}) = \mathfrak{C}'$.)

An action on a general basic object $(\mathcal{F}_0, (W_0, E_0))$ is an isomorphism of general basic objects of $(\mathcal{F}_0, (W_0, E_0))$ onto itself.

Or equivalently, we can define an action on a general basic object $(\mathcal{F}_0, (W_0, E_0))$ in the following way.

Let $(\mathcal{F}_0, (W_0, E_0))$ be a general basic object over $(F_0, (W_0, E_0))$, with a d-dimensional structure given by the charts $\{(\widetilde{W_0^\lambda}, (\mathfrak{a}_0^\lambda, b^\lambda), \widetilde{E_0^\lambda})\}$ of basic objects of dimension d, with the collection $\mathfrak{C}$ of transformations and smooth morphisms represented by $(\mathcal{F}_0, (W_0, E_0))$.

Let θ be an action on the pair (W_0, E_0).

Remark that the new charts $\{({}^\theta\widetilde{W_0^\lambda}, ({}^\theta\mathfrak{a}_0^\lambda, {}^\theta b^\lambda), {}^\theta\widetilde{E_0^\lambda})\}$, where

$$\begin{cases} {}^\theta\widetilde{W_0^\lambda} = \theta(\widetilde{W_0^\lambda}) \\ {}^\theta\mathfrak{a}_0^\lambda = \theta_*(\mathfrak{a}_0^\lambda) \\ {}^\theta b^\lambda = b^\lambda \\ {}^\theta\widetilde{E_0^\lambda} = \theta(\widetilde{E_0^\lambda}), \end{cases}$$

defines a general basic object $({}^\theta\mathcal{F}_0, ({}^\theta W_0 = W_0, {}^\theta E_0 = E_0))$ over $({}^\theta F_0 = \theta(F_0), (W_0, E_0))$ with the collection ${}^\theta\mathfrak{C}$ of transformations and restrictions represented by $({}^\theta\mathcal{F}_0, (W_0, E_0))$.

We say that θ is an action on the general basic object $(\mathcal{F}_0,(W_0,E_0))$ if

$$F_0 = {}^\theta F_0 \quad \& \quad \mathfrak{C} = {}^\theta\mathfrak{C}.$$

That is to say, θ is an action if $F_0 = {}^\theta F_0$ and if a necessary and sufficient condition for a sequence of pairs with specified basic object

$$(F_0,(W_0,E_0)) \stackrel{\pi_1}{\longleftarrow} (F_1,(W_1,E_1)) \stackrel{\pi_2}{\longleftarrow} \cdots \stackrel{\pi_{k-1}}{\longleftarrow} (F_{k-1},(W_{k-1},E_{k-1})) \stackrel{\pi_k}{\longleftarrow} (F_k,(W_k,E_k))$$

to be in the collection $\mathfrak{C}$ is for its θ-counterpart (the sequence below which makes an obvious commutative diagram with the sequence above having the vertical arrows being θ)

$$\begin{array}{l} ({}^\theta F_0,({}^\theta W_0,{}^\theta E_0)) \stackrel{{}^\theta\pi_1}{\longleftarrow} ({}^\theta F_1,({}^\theta W_1,{}^\theta E_1)) \stackrel{{}^\theta\pi_2}{\longleftarrow} \cdots \stackrel{{}^\theta\pi_{k-1}}{\longleftarrow} ({}^\theta F_{k-1},({}^\theta W_{k-1},{}^\theta E_{k-1})) \stackrel{{}^\theta\pi_k}{\longleftarrow} ({}^\theta F_k,({}^\theta W_k,{}^\theta E_k)) \\ \qquad\qquad \Big\| \\ (F_0,(W_0,E_0)) \end{array}$$

to be in the collection $\mathfrak{C}$.

We emphasize that, in order for θ to be an action on the general basic object, we are NOT requiring the charts $\{(\widetilde{W_0^\lambda},(\mathfrak{a}_0^\lambda,b^\lambda),\widetilde{E_0^\lambda})\}$ to coincide with the new charts $\{({}^\theta\widetilde{W_0^\lambda},({}^\theta\mathfrak{a}_0^\lambda,{}^\theta b^\lambda),{}^\theta\widetilde{E_0^\lambda})\}$ but that we are requiring the collection $\mathfrak{C}$ to coincide with the new collection ${}^\theta\mathfrak{C}$.

Definition 8-6 (Equivariant sequence of general basic objects). *Let*

$$(\mathcal{F}_0,(W_0,E_0)) \leftarrow \cdots \leftarrow (\mathcal{F}_k,(W_k,E_k))$$

be a sequence of transformations and smooth morphisms of general basic objects. (Remark that the above is nothing but a notational convention (cf. Note 4-3) expressing a sequence of transformations and smooth morphisms

$$(F_0,(W_0,E_0)) \leftarrow \cdots \leftarrow (F_k,(W_k,E_k))$$

in the collection $\mathfrak{C}$ represented by $(\mathcal{F}_0,(W_0,E_0))$. We say that the sequence is θ-equivariant, given an action θ on the general basic object $(\mathcal{F}_0,(W_0,E_0))$, if inductively for $i=1,...,k$ we have a commutative diagram

$$\begin{array}{ccc} (\mathcal{F}_{i-1},(W_{i-1},E_{i-1})) & \stackrel{\pi_i}{\longleftarrow} & (\mathcal{F}_i,(W_i,E_i)) \\ {\scriptstyle\theta}\Big\downarrow & & {\scriptstyle\theta}\Big\downarrow \\ (\mathcal{F}_{i-1},(W_{i-1},E_{i-1})) & \stackrel{\pi_i}{\longleftarrow} & (\mathcal{F}_i,(W_i,E_i)) \end{array}$$

i.e., the action on $(\mathcal{F}_{i-1},(W_{i-1},E_{i-1}))$ induced by θ on $(\mathcal{F}_0,(W_0,E_0))$ lifts to the action on $(\mathcal{F}_i,(W_i,E_i))$. (Remark that it is simply equivalent to requiring that the action on the pair (W_{i-1},E_{i-1}) induced by θ on (W_0,E_0) lifts to the action on (W_i,E_i) and $\theta(F_i)=F_i$.)

We note that if the sequence is θ-equivariant, then

$$\begin{cases} \theta(Y_{i-1}) = Y_{i-1}^\theta = Y_{i-1} \\ \textit{whenever } \pi_i \textit{ is a transformation with center } Y_{i-1} \subset F_{i-1} \subset W_{i-1}. \end{cases}$$

We say that the sequence is equivariant if it is θ-equivariant for any *action θ on the general basic object $(\mathcal{F}_0,(W_0,E_0))$.*

Definition 8-7 (Equivariant resolution of singularities of a general basic object). *Let*

$$(\mathcal{F}_0, (W_0, E_0)) \leftarrow \cdots \leftarrow (\mathcal{F}_k, (W_k, E_k))$$

be a sequence of transformations representing resolution of singularities of the general basic object $(\mathcal{F}_0, (W_0, E_0))$ *(i.e.,* $F_k = \emptyset$*).*

We say that the resolution of singularities is θ*-equivariant, given an action* θ *on* $(\mathcal{F}_0, (W_0, E_0))$*, if the sequence is* θ*-equivariant.*

We say that the resolution of singularities is equivariant if the sequence is equivariant.

Remark 8-8.

There is some confusion concerning the definitions of an action and an equivariant sequence of general basic objects in the original paper "A course on constructive desingularization and equivariance" by Encinas and Villamayor.

In Definition 6.20 (of the paper) they define:

We say an automorphism $\theta : W_0 \overset{\sim}{\to} W_0$ acts on the general basic object $(\mathcal{F}_0, (W_0, E_0))$ if:

(a) θ acts on the pair (W_0, E_0) and $\theta(F_0) = F_0$, and

(b) for any sequence in $\mathfrak{C}$

$$(F_0, (W_0, E_0)) \leftarrow \cdots \leftarrow (F_k, (W_k, E_k))$$

which is θ-equivariant in the sense that

$$\begin{cases} \theta(Y_{i-1}) = Y_{i-1} \\ \text{whenever } \pi_i \text{ is a transformation with center } Y_{i-1} \subset F_{i-1} \subset W_{i-1}, \end{cases}$$

we have

$$\theta(F_i) = F_i \text{ for } i = 0, ..., k.$$

Their definition is clearly different from ours. With their definition of an "action" on a general basic object, one has trouble, e.g., in proving $\mathrm{ord}_0(x_0) = \mathrm{ord}_0(\theta(x_0))$ for an action on a general basic object $(\mathcal{F}_0, (W_0, E_0))$ and $x_0 \in F_0 \subset W_0$. The sequence we construct in Hironaka's trick is NOT θ-equivariant (in their sense as above) and their definition of the action does not provide any information. They tried to compensate for this calamity of their definition by looking at x_0 and $\theta(x_0)$ simultaneously in Hironaka's trick (in a vain attempt to make the sequence equivariant) and claiming in their proof of Proposition 7.4 that "$\theta \times Id$ acts on (W_1, E_1) interchanging the components of $x_0 \times \mathbb{A}^1$ and $\theta(x_0) \times \mathbb{A}^1$ and interchanging $(x_0, 0)$ and $(\theta(x_0), 0)$", which is sheer nonsense.

It is very clear, however, from the context of the paper that what Encinas and Villamayor really mean is the definition(s) that we give here in these notes and that the discrepancies mentioned above should be considered mere "typos" in the paper.

Lemma 8-9 (Invariance of key invariants for basic objects under action). *Let*

$$\begin{aligned}(W,(J,b),E) = (W_0,(J_0,b),E_0) &\stackrel{\pi_1}{\leftarrow} (W_1,(J_1,b),E_1) \stackrel{\pi_2}{\leftarrow} \cdots \\ (W_{i-1},(J_{i-1},b),E_{i-1}) &\stackrel{\pi_i}{\leftarrow} (W_i,(J_i,b),E_i) \\ \cdots \stackrel{\pi_{k-1}}{\leftarrow} (W_{k-1},(J_{k-1},b),E_{k-1}) &\stackrel{\pi_k}{\leftarrow} (W_k,(J_k,b),E_k)\end{aligned}$$

be a sequence of transformations and smooth morphisms of basic objects.

Let θ be a action on the basic object $(W,(J,b),E)$.

Let

$$\begin{aligned}(W_0,(J_0,b),E_0) = ({}^\theta W_0,({}^\theta J_0,b),{}^\theta E_0) &\stackrel{{}^\theta\pi_1}{\leftarrow} ({}^\theta W_1,({}^\theta J_1,b),{}^\theta E_1) \stackrel{{}^\theta\pi_2}{\leftarrow} \cdots \\ ({}^\theta W_{i-1},({}^\theta J_{i-1},b),{}^\theta E_{i-1}) &\stackrel{{}^\theta\pi_i}{\leftarrow} ({}^\theta W_i,({}^\theta J_i,b),{}^\theta E_i) \\ \cdots \stackrel{{}^\theta\pi_{k-1}}{\leftarrow} ({}^\theta W_{k-1},({}^\theta J_{k-1},b),{}^\theta E_{k-1}) &\stackrel{{}^\theta\pi_k}{\leftarrow} ({}^\theta W_k,({}^\theta J_k,b),{}^\theta E_k)\end{aligned}$$

be the θ-counterpart of the sequence.

Then for $\xi_k \in \mathrm{Sing}(J_k,b)$ and $\theta(\xi_k) = {}^\theta\xi_k \in \theta(\mathrm{Sing}(J_k,b)) = \mathrm{Sing}({}^\theta J_k,b)$ we have

$$\begin{cases} \mathrm{ord}_k(\xi_k) = {}^\theta\mathrm{ord}_k({}^\theta\xi_k), \\ w\text{-}\mathrm{ord}_k(\xi_k) = {}^\theta w\text{-}\mathrm{ord}_k({}^\theta\xi_k) \quad \& \quad \theta(\underline{\mathrm{Max}}\ w\text{-}\mathrm{ord}_k) = \underline{\mathrm{Max}}\ {}^\theta w\text{-}\mathrm{ord}_k, \\ \Gamma_k(\xi_k) = {}^\theta\Gamma_k({}^\theta\xi_k) \qquad \& \quad \theta(\underline{\mathrm{Max}}\ \Gamma_k) = \underline{\mathrm{Max}}\ {}^\theta\Gamma_k \\ \qquad \text{if } \max\ w\text{-}\mathrm{ord}_k = 0 \text{ and hence we may regard} \\ \qquad (W_k,(J_k,b),E_k) \text{ and } ({}^\theta W_k,({}^\theta J_k,b),{}^\theta E_k) \text{ as monomial basic objects} \\ \qquad \text{in open neighborhoods of } \mathrm{Sing}(J_k,b) \text{ and } \theta(\mathrm{Sing}(J_k,b)) = \mathrm{Sing}({}^\theta J_k,b) \text{ respectively,} \\ t_k(\xi_k) = {}^\theta t_k({}^\theta\xi_k) \qquad \& \quad \theta(\underline{\mathrm{Max}}\ t_k) = \underline{\mathrm{Max}}\ {}^\theta t_k \\ \qquad \text{if the sequence satisfies condition } (\heartsuit), \end{cases}$$

where ${}^\theta\mathrm{ord}_k, {}^\theta w\text{-}\mathrm{ord}_k, {}^\theta\Gamma_k, {}^\theta t_k$ are the ord, w-ord, Γ, t-invariants on $({}^\theta W_k,({}^\theta J_k,b),{}^\theta E_k)$ (defined with respect to the θ-counterpart of the sequence).

If the sequence is θ-equivariant, then θ lifts to an action on $(W_k,(J_k,b),E_k)$ and we have

$$\begin{cases} \mathrm{ord}_k(\xi_k) = \mathrm{ord}_k({}^\theta\xi_k), \\ w\text{-}\mathrm{ord}_k(\xi_k) = w\text{-}\mathrm{ord}_k({}^\theta\xi_k) \quad \& \quad \theta(\underline{\mathrm{Max}}\ w\text{-}\mathrm{ord}_k) = \underline{\mathrm{Max}}\ w\text{-}\mathrm{ord}_k, \\ \Gamma_k(\xi_k) = \Gamma_k({}^\theta\xi_k) \qquad \& \quad \theta(\underline{\mathrm{Max}}\ \Gamma_k) = \underline{\mathrm{Max}}\ \Gamma_k \\ \qquad \text{if } \max\ w\text{-}\mathrm{ord}_k = 0 \text{ and hence we may regard} \\ \qquad (W_k,(J_k,b),E_k) \text{ as a monomial basic object} \\ \qquad \text{in an open neighborhood of } \mathrm{Sing}(J_k,b) \\ t_k(\xi_k) = t_k({}^\theta\xi_k) \qquad \& \quad \theta(\underline{\mathrm{Max}}\ t_k) = \underline{\mathrm{Max}}\ t_k \\ \qquad \text{if the sequence satisfies condition } (\heartsuit). \end{cases}$$

Proof.

The proof is obvious from the definition. Note that stability (equivariance) of the key invariants under isomorphism of basic objects can be stated and proved in a similar manner.

Lemma 8-10 (Invariance of key invariants for general basic objects under action). *Let*

$$(\mathcal{F}_0, (W_0, E_0)) \leftarrow \cdots \leftarrow (\mathcal{F}_k, (W_k, E_k))$$

be a sequence of transformations and smooth morphisms of general basic objects.

Let θ be a action on the general basic object $(\mathcal{F}_0, (W_0, E_0))$ with a d-dimensional structure.

Let

$$({}^\theta\mathcal{F}_0, ({}^\theta W_0, {}^\theta E_0)) \leftarrow \cdots \leftarrow ({}^\theta\mathcal{F}_k, ({}^\theta W_k, {}^\theta E_k))$$

be the θ-counterpart of the sequence.

Then for $\xi_k \in F_k$ and $\theta(\xi_k) = {}^\theta\xi_k \in \theta(F_k) = {}^\theta F_k$ we have

$$\begin{cases} \mathrm{ord}_k(\xi_k) = {}^\theta\mathrm{ord}_k({}^\theta\xi_k), \\ w\text{-}\mathrm{ord}_k(\xi_k) = {}^\theta w\text{-}\mathrm{ord}_k({}^\theta\xi_k) \quad \& \quad \theta(\underline{\mathrm{Max}}\ w\text{-}\mathrm{ord}_k) = \underline{\mathrm{Max}}\ {}^\theta w\text{-}\mathrm{ord}_k, \\ \Gamma_k(\xi_k) = {}^\theta\Gamma_k({}^\theta\xi_k) \qquad \& \quad \theta(\underline{\mathrm{Max}}\ \Gamma_k) = \underline{\mathrm{Max}}\ {}^\theta\Gamma_k \\ \qquad \textit{if } \max\ w\text{-}\mathrm{ord}_k = 0 \\ t_k(\xi_k) = {}^\theta t_k({}^\theta\xi_k) \qquad \& \quad \theta(\underline{\mathrm{Max}}\ t_k) = \underline{\mathrm{Max}}\ {}^\theta t_k \\ \qquad \textit{if the sequence satisfies condition } (\heartsuit), \end{cases}$$

where ${}^\theta ord_k, {}^\theta w\text{-}ord_k, {}^\theta\Gamma_k, {}^\theta t_k$ are the $ord, w\text{-}ord, \Gamma, t$-invariants on $({}^\theta\mathcal{F}_k, ({}^\theta W_k, {}^\theta E_k))$ (defined with respect to the θ-counterpart of the sequence).

If the sequence is θ-equivariant, then θ lifts to an action on $(\mathcal{F}_k, (W_k, E_k))$ with a $\dim W_k - \dim W_0 + d$-dimensional structure and we have

$$\begin{cases} \mathrm{ord}_k(\xi_k) = \mathrm{ord}_k({}^\theta\xi_k), \\ w\text{-}\mathrm{ord}_k(\xi_k) = w\text{-}\mathrm{ord}_k({}^\theta\xi_k) \quad \& \quad \theta(\underline{\mathrm{Max}}\ w\text{-}ord_k) = \underline{\mathrm{Max}}\ w\text{-}ord_k, \\ \Gamma_k(\xi_k) = \Gamma_k({}^\theta\xi_k) \qquad \& \quad \theta(\underline{\mathrm{Max}}\ \Gamma_k) = \underline{\mathrm{Max}}\ \Gamma_k \\ \qquad \textit{if } \max\ w\text{-}\mathrm{ord}_k = 0 \\ t_k(\xi_k) = t_k({}^\theta\xi_k) \qquad \& \quad \theta(\underline{\mathrm{Max}}\ t_k) = \underline{\mathrm{Max}}\ t_k \\ \qquad \textit{if the sequence satisfies condition } (\heartsuit). \end{cases}$$

Proof.

If $\xi_k \in F_k \cap W_k^\lambda = \mathrm{Sing}(\mathfrak{a}_k^\lambda, b^\lambda)$ is a point of a chart $(\widetilde{W_k^\lambda}, (\mathfrak{a}_k^\lambda, b^\lambda), \widetilde{E_k^\lambda})$, then ${}^\theta\xi_k \in {}^\theta F_k \cap {}^\theta(W_k^\lambda) = \mathrm{Sing}({}^\theta\mathfrak{a}_k^\lambda, {}^\theta b^\lambda)$ is a point of the chart $({}^\theta\widetilde{W_k^\lambda}, ({}^\theta\mathfrak{a}_k^\lambda, {}^\theta b^\lambda), {}^\theta\widetilde{E_k^\lambda})$. Therefore, the above assertions are easy consequences of Definition-Proposition 4-5 and Lemma 8-9. Note that stability (equivariance) of the key invariants under isomorphism of general basic objects can be stated and proved in a similar manner.

Proposition 8-11 (Equivariance of the inductive algorithm). *Let*

$$(\mathcal{F}_0, (W_0, E_0)) \leftarrow \cdots \leftarrow (\mathcal{F}_k, (W_k, E_k))$$

be the sequence of transformations of general basic objects, obtained via the inductive algorithm of Theorem 5-1, representing resolution of singularities of a general basic object $(\mathcal{F}_0, (W_0, E_0))$ with a d-dimensional structure.

Let θ be an action on the general basic object $(\mathcal{F}_0,(W_0,E_0))$ with a d-dimensional structure.

Then the sequence is θ-equivariant.

In particular, the sequence is equivaraint.

Proof.

We have only to prove that, having a θ-equivariant sequence,

$$(\mathcal{F}_0,(W_0,E_0)) \leftarrow \cdots \leftarrow (\mathcal{F}_k,W_k,E_k)),$$

the extension given by the process of the inductive algorithm of Theorem 5-1 is also θ-equivariant.

P1: The sequence already represents resolution of singularities and there is no need for the extension. The original sequence is θ-equivariant by assumption.

P2: The extension of the sequence of transformations we create via Corollary 4-9 is θ-equivariant, since the centers for the transformations

$$(\mathcal{F}_i,(W_i,E_i)) \leftarrow (\mathcal{F}_{i+1},(W_{i+1},E_{i+1})) \text{ for } i \geq k$$

are θ-invariant., i.e., $\theta(\underline{\text{Max}}\ \Gamma_i) = \underline{\text{Max}}\ \Gamma_i$ by Lemma 8-9.

P3: We deal with the case of possibility **P3** in the following.

Case A: We have $\theta(R(1)(\underline{\text{Max}}\ t_k)) = R(1)(\underline{\text{Max}}\ t_k)$, since $\theta(\underline{\text{Max}}\ t_k) = \underline{\text{Max}}\ t_k$ by Lemma 8-9.

Thus the extended sequence by adding $(\mathcal{F}_k,(W_k,E_k)) \leftarrow (\mathcal{F}_{k+1},(W_{k+1},E_{k+1}))$ is θ-equivariant.

Case B: Note first that θ induces an action on the general basic object $(\mathcal{F}_k,(W_k,E_k))$ (since the sequence is θ-equivariant) and that by the property $\theta(\underline{\text{Max}}\ t_k) = \underline{\text{Max}}\ {}^{\theta}t_k)$ of Lemma 8-10 (not only referring to the case of the index k but also to the further extension) θ induces an action on the general basic object $(\mathcal{G}_k,(W_k,E''_k))$ via Lemma 5-4. (Recall that the properties (α) and (β) provide a characterization of the general basic object $(\mathcal{G}_k,(W_k,E''_k))$. See the conclusion of the proof of Theorem 5-1 for the assertions in **Case B** under possibility **P3**.) The sequence of transformations, constructed via the inductive algorithm and representing resolution of singularities of $(\mathcal{G}_k,(W_k,E''_k))$, is θ-equivariant by induction on the dimension of the structure d. (We leave the proof of θ-equivariance in the case $d = 1$ to the reader as an exercise.) Therefore, the extended sequence adding

$$(\mathcal{F}_k,(W_k,E_k)) \leftarrow \cdots \leftarrow (\mathcal{F}_{k+N},(W_{k+N},E_{k+N}))$$

is also θ-equivariant.

This completes the proof of Proposition 8-11.

Corollary 8-12 (Equivaraint resolution of singularities of a general basic object). *Let $(\mathcal{F}_0,(W_0,E_0))$ be a general basic object with a d-dimensional structure.*

Then there exists equivariant resolution of singularities of $(\mathcal{F}_0,(W_0,E_0))$

$$(\mathcal{F}_0,(W_0,E_0)) \leftarrow \cdots \leftarrow (\mathcal{F}_k,(W_k,E_k))$$

satisfying condition $(\heartsuit')$

$(\heartsuit')$ $Y_{i-1} \subset \underline{\text{Max}}\ t_{i-1} \subset \underline{\text{Max}}\ w\text{-ord}_{i-1}$ *if* $\max\ w\text{-ord}_{i-1} > 0$ *for* $i = 1, ..., k$.

Proof.

We only need to check that the sequence given by the inductive algorithm to represent resolution of singularities, which satisfies condition $(\heartsuit')$ by construction, is equivariant. This is exactly the content of Proposition 8-11.

Corollary 8-13 (Equivaraint resolution of singularities of a basic object). *Let* $(W_0, (J_0, b), E_0)$ *be a basic object. Then there exists equivariant resolution of singularities of* $(W_0, (J_0, b), E_0)$

$$(W_0, (J_0, b), E_0) \leftarrow \cdots \leftarrow (W_k, (J_k, b), E_k)$$

satisfying condition $(\heartsuit')$

$(\heartsuit')$ $Y_{i-1} \subset \underline{\text{Max}}\ t_{i-1} \subset \underline{\text{Max}}\ w\text{-ord}_{i-1}$ *if* $\max\ w\text{-ord}_{i-1} > 0$ *for* $i = 1, ..., k$.

Proof.

This is a direct consequence of Corollary 8-12 and Remark 4-2 (ii).

Corollary 8-14 (Resolution of singularities of a general basic object (resp. basic object) over any field k of characteristic zero). *Let* $(\mathcal{F}_0, (W_0, E_0))$ *(resp.* $(W_0, (J_0, b), E_0)$*) be a general basic object with a d-dimensional structure (resp. basic object) defined over a field k which is of characteristic zero but may not be algebraically closed. Then there exists resolution of singularities, satisfying condition* $(\heartsuit')$*,*

$$\begin{gathered}(\mathcal{F}_0, (W_0, E_0)) \leftarrow \cdots \leftarrow (\mathcal{F}_k, (W_k, E_k)) \\ (\textit{resp. } (W_0, (J_0, b), E_0) \leftarrow \cdots \leftarrow (W_k, (J_k, b), E_k))\end{gathered}$$

which is defined over k.

Proof.

Firstly remark that a general basic object $(\mathcal{F}_0, (W_0, E_0))$ is defined over k with a d-dimensional structure if, by definition, it has an open covering $\{W_0^\lambda\}_{\lambda \in \Lambda}$ with charts $\{\widetilde{W_0^\lambda}, (\mathfrak{a}_0^\lambda, b^\lambda), \widetilde{E_0^\lambda})\}_{\lambda \in \Lambda}$ which are defined over k and that by $(\mathcal{F}_0, (W_0, E_0)) \times \text{Spec}\ \overline{k}$ we mean the general basic object having the open covering $\{W_0^\lambda \times \text{Spec}\ \overline{k}\}_{\lambda \in \Lambda}$ with charts $\{\widetilde{W_0^\lambda}, (\mathfrak{a}_0^\lambda, b^\lambda), \widetilde{E_0^\lambda}) \times \text{Spec}\ \overline{k}\}_{\lambda \in \Lambda}$. It should be warned, however, that the collection (of sequences of transformations and smooth morphisms of pairs with specified closed subsets) represented by $(\mathcal{F}_0, (W_0, E_0)) \times \text{Spec}\ \overline{k}$ contains more than those obtained by taking the Cartesian product of the sequences defined over k in the collection represented by the original $(\mathcal{F}_0, (W_0, E_0))$ with Spec $\overline{k}$.

Secondly note that, since we have been assuming so far that the ambient space W_0 to be irreducible, we need to generalize the theory to the case where W_0 may be reducible and of pure dimension, as $W_0 \times \text{Spec}\ \overline{k}$ may be. This generalization can be made without any change in the arguments.

Thirdly note that the Galois group $\mathrm{Gal}(\overline{k}/k)$ acts on the general basic object $(\mathcal{F}_0, (W_0, E_0)) \times \mathrm{Spec}\ \overline{k}$ (resp. $(W_0, (J_0, b), E_0) \times \mathrm{Spec}\ \overline{k}$) in the sense of Definition 8-5 (resp. Definition 8-1).

We construct a sequence of transformations representing resolution of singularities of $(\mathcal{F}_0, (W_0, E_0)) \times \mathrm{Spec}\ \overline{k}$ (resp. $(W_0, (J_0, b), E_0) \times \mathrm{Spec}\ \overline{k}$) via the inductive algorithm of Theorem 5-1. Since the sequence is equivariant under the action of the Galois group $\mathrm{Gal}(\overline{k}/k)$ by Proposition 8-11, the centers $\overline{Y_{i-1}}$ are defined over k, i.e., $\overline{Y_{i-1}} = Y_{i-1} \times \mathrm{Spec}\ \overline{k}$ for some closed subscheme $Y_{i-1} \subset W_{i-1}$ defined over k and the sequence of transformations with centers Y_{i-1} provides resolution of singularities of $(\mathcal{F}_0, (W_0, E_0))$ (resp. $(W_0, (J_0, b), E_0)$ over k.

Corollary 8-15 (Embedded resolution of singularities over any field k of characteristic zero). *Let $X \subset W$ be a variety, embedded as a closed subscheme (defined over k) of another variety W smooth over a field k which is of characteristic zero but may not be algebraically closed. Then there exists a sequence defined over k of blowups representing embedded resolution of singularities of $X \subset W$, which is equivariant in the sense that for any automorphism $\theta : W \xrightarrow{\sim} W$ with $\theta(X) = X$, we have all the centers of the blowups being θ-invariant.*

Proof.

This is an easy consequnce of Corollary 8-14 combined with our construction of embedded resolution of singularities presented in Chapter 7.

Remark 8-16 (Equivariance of the sequence representing resolution of singularities in Corollary 8-14 or Corollary 8-15).

Let θ be an action on the general basic object $(\mathcal{F}_0, (W_0, E_0))$ with a d-dimensional structure (defined over k). We claim that the sequence constructed via the inductive algorithm of Theorem 5-1 is θ-equivarinat, no matter whether θ is over k or not. (Recall that θ is an isomorphism as abstract varieties, satisfying certain conditions as described in Definition 8-5, and that it is not necessarily over the base field k.)

If $\theta : (\mathcal{F}_0, (W_0, E_0)) \xrightarrow{\sim} (\mathcal{F}_0, (W_0, E_0))$ is over k, then it extends to an action $\theta \times Id : (\mathcal{F}_0, (W_0, E_0)) \times \mathrm{Spec}\ \overline{k} \xrightarrow{\sim} (\mathcal{F}_0, (W_0, E_0)) \times \mathrm{Spec}\ \overline{k}$. Therefore, equivariance of θ, i.e., $\theta(Y_{i-1}) = Y_{i-1}$ follows that of $\theta \times Id$.

However, if $\theta : (\mathcal{F}_0, (W_0, E_0)) \xrightarrow{\sim} (\mathcal{F}_0, (W_0, E_0))$ is not over k, there is no obvious way that θ extends to an action on $(\mathcal{F}_0, (W_0, E_0)) \times \mathrm{Spec}$. In other words, there is no obvious way to reduce the equivariance over k to that over $\overline{k}$ taking the Cartesion product $\times_{\mathrm{Spec}\ k} \mathrm{Spec}\ \overline{k}$.

We just remark that the entire theory up to Chapter 7 and Chapter 8 can be developed over any field of characteristic zero, without assuming that k is algebraically closed and that θ-equivariance of the inductive algorithm, given any action whether it is defined over k or not, goes verbatim as in the proof of Proposition 8-11. The essential point is that the invariants ord, w-ord, t, and Γ (of the original general basic object and of the auxiliary general basic objects appearing in the inductive process (See the characterization via properties (α) and (β).)), which determine the inductive algorithm, depend only on the structure as abstract varieties and not on the structure over k and hence are preserved under any isomorphism as abstract varieties whether it is over k or not. (We note that the only place where we use the assumption of the base field k being algebraically closed is the definition of

the extension Δ by making explicit the partial derivatives $\frac{\partial}{\partial x_i}$ via isomorphism $\widehat{\mathcal{O}_{W,p}} \cong k[[x_1, ..., x_d]]$ for a choice of the system of regular parameters $(x_1, ..., x_d)$ of m_p. One can do this without looking at the isomorphism, and define the partial derivatives and extension over any field k of characteristic zero. Then the rest of the argument goes without any change. The details are left to the reader as an exercise.)

We finish this section stating the stability of our inductive algorithm (and hence that of resolution process constructed via the inductive algorithm) under smooth morphisms.

Theorem 8-17 (Stability of the inductive algorithm under smooth morphism). *Let $\theta : (\mathcal{F}_0^\theta, (W_0^\theta, E_0^\theta)) \to (\mathcal{F}_0, (W_0, E_0))$ be a smooth morphism of general basic objects with a $(d^\theta = \dim W_0^\theta - \dim W_0 + d)$-structure and a d-dimensional structure, respectively. Let*

$$(\mathcal{F}_0, (W_0, E_0)) \leftarrow \cdots \leftarrow (\mathcal{F}_k, (W_k, E_k))$$

be a sequence of transformations of general basic objects satisfying condition ($\heartsuit$) *and*

$$(\mathcal{F}_0^\theta, (W_0^\theta, E_0^\theta)) \leftarrow \cdots \leftarrow (\mathcal{F}_k^\theta, (W_k^\theta, E_k^\theta))$$

be the sequence obtained by taking the Cartesian product of the first with the smooth morphism θ.

Then the second is a sequence of transformations of general basic objects satisfing condition ($\heartsuit$)*, and the extension of the second sequence described by the inductive algorithm is exactly the sequence*

$$(\mathcal{F}_k^\theta, (W_k^\theta, E_k^\theta)) \leftarrow \cdots \leftarrow (\mathcal{F}_{k+N}^\theta, (W_{k+N}^\theta, E_{k+N}^\theta))$$

obtained by taking the Cartesian product of the smooth morphism with the extension of the first sequence described by the inductive algorithm

$$(\mathcal{F}_k, (W_k, E_k)) \leftarrow \cdots \leftarrow (\mathcal{F}_{k+N}, (W_{k+N}, E_{k+N}))$$

(and by ignoring the trivial transformations whenever the pull-backs of the centers are empty).

We say that the inductive algorithm is stable under smooth morphisms of general basic objects.

In particular, the sequence representing resolution of singularities of a (general) basic object constructed via the inductive algorithm is also stable under smooth morphisms.

Proof.

One can prove the invariance (stability) of the key invariants under smooth morphisms in an identical manner to the one for proving the invariance of the key invariants under actions. Then the rest of the proof goes almost verbatim as that of Proposition 8-11. The details are left to the reader as an exercise.

We remark that the inductive algorithm is stable under any field extensions (of characteristic zero but may not be of finite type), which can be proved in an identical manner.

CHAPTER 9. INVARIANTS REVISITED

In this chapter, we construct the invariant f^d, based upon the key invariants (w-ord, Γ and t), associated to a (sequence of transformations of) general basic object(s) with a d-dimensional structure so that the centers of blowups in our inductive algorithm for resolution of singularities are exactly the loci where the invariant f^d attains its maximum. Since the invariant f^d is easily seen to be stable under any action (or more generally any smooth morphism), this will provide another easy proof for the equivariance (stability under smooth morphisms) of the inductive algorithm.

Our invariant f^d is slightly different from the one given in the paper "A course on constructive desingularization and equivariance" by Encinas and Villamayor, where their invariant uses such global information as the global maximum of the t-invariant and hence it is not stable under open immersions, much less so under general smooth morphisms.

Definition-Construction 9-1 (Invariant $\mathbf{f^d}$).

We define and construct the invariant f^d by induction on the dimension d of the structure of a general basic object.

Let

$$(\mathcal{F}_0, (W_0, E_0)) \leftarrow \cdots \leftarrow (\mathcal{F}_k, (W_k, E_k))$$

be a sequence of transformations of general basic objects satisfying condition ($\heartsuit$)

$$(\heartsuit)\ Y_{i-1} \subset \underline{\mathrm{Max}}\ w\text{-}\mathrm{ord}_{i-1} \subset F_{i-1} \text{ for } i = 1, ..., k.$$

$\boxed{\text{Case}: \ d = 1.}$

When the dimension d of the structure of the general basic objects is equal to 1, we define

$$f^1_k : F_k \to \{\{0\} \times \Gamma^1\} \sqcup \{\mathbf{W}_{>0} \times \mathbf{T} \times \{\infty\}\}$$

in the following way: for $p \in F_k$

$$f^1_k(p) = \begin{cases} (w\text{-}\mathrm{ord}_k(p), \Gamma_k(p)) & \in \{0\} \times \Gamma^1 \quad\quad\quad \text{if} \quad w\text{-}\mathrm{ord}_k(p) = 0 \\ (w\text{-}\mathrm{ord}_k(p), t_k(p), \infty) & \in \mathbf{W}_{>0} \times \mathbf{T} \times \{\infty\} \ \text{if} \quad w\text{-}\mathrm{ord}_k(p) > 0 \end{cases}$$

with

$$\begin{cases} w\text{-}\mathrm{ord}_k(p) \in \{0\} \sqcup \mathbf{W}_{>0} = \mathbf{W} = \dfrac{1}{c!}\mathbb{Z}_{\geq 0} \\ t_k(p) \in \mathbf{T} = \dfrac{1}{c!}\mathbb{Z}_{\geq 0} \times \mathbb{Z}_{\geq 0} \\ \Gamma_k(p) \in \Gamma^1 = (\mathbb{Z}_{\geq -1} \times \dfrac{1}{c!}\mathbb{Z}_{\geq 0} \times \mathbb{Z}^1_{\geq 0}). \end{cases}$$

Note that when w-$\mathrm{ord}_k(p) = 0$, by upper semi-continuity, w-ord_k is zero in a neighborhood of p and hence that $\Gamma_k(p)$ is well-defined (cf. Definitin-Proposition 4-5 (iv)).

We refer the reader to (GB-3) of Definition 4-1 of general basic objects for the meaning of the number c.

Note that we give the obvious lexicographical order (which is a total order) to the set $I_1 = \{\{0\} \times \Gamma^1\} \sqcup \{\mathbf{W}_{>0} \times \mathbf{T} \times \{\infty\}\}$, induced from the lexicographical orders on $\mathbf{W}$, Γ^1, and $\mathbf{T}$.

(The superfluous-looking $\{\infty\}$ in the case $w\text{-ord}_k(p) > 0$ is added to make the invariant f^d stable under smooth morphisms.)

$\boxed{\text{Case : } d = d \text{ based upon Case : } d = d-1 \text{ by induction.}}$

Suppose we have already defined the invariant f^{d-1} (with values in a totally ordered set I_{d-1}) associated to a (sequence of) general basic object(s) with a $(d-1)$-dimensional structure.

We define

$$f_k^d : F_k \to I_d = \{\{0\} \times \Gamma^d\} \sqcup \{\mathbf{W}_{>0} \times \mathbf{T} \times \{\infty\}\} \sqcup \{\mathbf{W}_{>0} \times \mathbf{T} \times \{0\} \times I_{d-1}\}$$

in the following way: for $p \in F_k$

$$f_k^d(p) = \begin{cases} (w\text{-ord}_k(p), \Gamma_k(p)) & \in \{0\} \times \Gamma^d \\ \qquad \text{if } w\text{-ord}_k(p) = 0 & \\ (w\text{-ord}_k(p), t_k(p), \infty) & \in \mathbf{W}_{>0} \times \mathbf{T} \times \{\infty\} \\ \qquad \text{if } w\text{-ord}_k > 0 \text{ and } R(1)(\{q \in F_k; t_k(q) = t_k(p)\})_p \neq \emptyset & \\ (w\text{-ord}_k(p), t_k(p), 0, {f''}_k^{d-1}(p)) & \in \{\mathbf{W}_{>0} \times \mathbf{T} \times \{0\} \times I_{d-1}\} \\ \qquad \text{if } w\text{-ord}_k > 0 \text{ and } R(1)(\{q \in F_k; t_k(q) = t_k(p)\})_p = \emptyset & \end{cases}$$

with

$$\begin{cases} w\text{-ord}_k(p) \in \{0\} \sqcup \mathbf{W}_{>0} = \mathbf{W} = \dfrac{1}{c!}\mathbb{Z}_{\geq 0} \\ t_k(p) \in \mathbf{T} = \dfrac{1}{c!}\mathbb{Z}_{\geq 0} \times \mathbb{Z}_{\geq 0} \\ \Gamma_k(p) \in \Gamma^d = (\mathbb{Z}_{\geq -d} \times \dfrac{1}{c!}\mathbb{Z}_{\geq 0} \times \mathbb{Z}_{\geq 0}^d), \end{cases}$$

where $R(1)(\{q \in F_k; t_k(q) = t_k(p)\})_p$ is the codimension one part, i.e., the $(d-1)$-dimensional part of the locus $\{q \in F_k; t_k(q) = t_k(p)\}$ passing through the point p.

The invariant ${f''}_k^{d-1} : G_{k,V} \to I_{d-1}$ is given by the inductional assumption, defined on the general basic object $(\mathcal{G}_{k,V}, (V, {E''}_{k,V}))$ over $(G_{k,V}, (V, {E''}_{k,V}))$ with a $(d-1)$-dimensional structure, constructed in the following way:

<u>Construction and charactreization of the general basic object $(\mathcal{G}_{k,V}, (V, {E''}_{k,V}))$</u>

Take an open neighborhood V of p such that

$$w\text{-ord}_k(p) = \max\{w\text{-ord}_k(q); q \in V \cap F_k\}$$
$$t_k(p) = \max\{t_k(q); q \in V \cap F_k\}.$$

In short, we follow the construction of $(\mathcal{G}_k, (W_k, E_k''))$ carried out in Lemma 5-3 and Lemma 5-4 (for the conclusion of the proof for the assertions in **Case B** under

possibility **$\underline{\mathbf{P\ 3}}$**), locally over V. The general basic object we construct represents the collection of sequences of transformations and smooth morphisms with the specifeid closed subsets being the loci $\underline{\mathrm{Max}}\, t$ locally over V.

We describe the construction more precisely in what follows.

Take the extension of the original sequence (only for the purpose of constructing the general basic object $(\mathcal{G}_k, (W_k, E''_k))$ and defining the invariant ${f''}_k^{d-1}$)

$$(\mathcal{F}_0, (W_0, E_0)) \leftarrow \cdots \leftarrow (\mathcal{F}_k, (W_k, E_k)) \leftarrow (\mathcal{F}_{k+1}, (W_{k+1}, E_{k+1}))$$

where

$$(\mathcal{F}_k, (W_k, E_k)) \leftarrow (\mathcal{F}_{k+1} = \mathcal{F}_k|_V, (W_{k+1} = V, E_{k+1} = E_k|_V))$$

is induced by the open immersion $V \hookrightarrow W_k$.

We construct a general basic object over $(G_{k,V}, (V, {E''}_{k,V})) = (\underline{\mathrm{Max}}\, t_{k+1}, (W_{k+1}, E^+_{k+1}))$, with a d-dimensional structure first, by specifying its charts of basic objects $\{(\widetilde{W''_{k+1}}^{\lambda}, (\widetilde{\mathfrak{a}''_{k+1}}^{\lambda}, b''^{\lambda}), \widetilde{E''_{k+1}}^{\lambda})\}$ of dimension d in the following way:

Let $\{(\widetilde{W^{\lambda}_{k+1}}, (\mathfrak{a}^{\lambda}_{k+1}, b^{\lambda}), \widetilde{E^{\lambda}_{k+1}})\}$ be the charts for the general basic objects $(\mathcal{F}_{k+1}, (W_{k+1}, E_{k+1}))$ arising from the sequence (cf. Note 4-3)

$$(F_0, (W_0, E_0)) \leftarrow \cdots \leftarrow (F_k, (W_k, E_k)) \leftarrow (\mathcal{F}_{k+1}, (W_{k+1}, E_{k+1})).$$

We take $\widetilde{W''_{k+1}}^{\lambda} = \widetilde{W^{\lambda}_{k+1}}$.

If $\widetilde{W''_{k+1}}^{\lambda} \cap \underline{\mathrm{Max}}\, t_{k+1} = \emptyset$, then we take the basic object $(\widetilde{W''_{k+1}}^{\lambda}, (\mathfrak{a}''^{\lambda}_{k+1}, b''^{\lambda}), \widetilde{E''_{k+1}}^{\lambda})$ to be

$$\begin{cases} \widetilde{W''_{k+1}}^{\lambda} = \widetilde{W^{\lambda}_{k+1}} \\ {\mathfrak{a}''_{k+1}}^{\lambda} = \mathcal{O}_{\widetilde{W''_{k+1}}^{\lambda}} \\ b''^{\lambda} = 1 \\ \widetilde{E''_{k+1}}^{\lambda} = \widetilde{E^{\lambda}_{k+1}}^{+} = E^+_{k+1} \cap \widetilde{W''_{k+1}}^{\lambda}. \end{cases}$$

If $\widetilde{W''_{k+1}}^{\lambda} \cap \underline{\mathrm{Max}}\, t_{k+1} \neq \emptyset$, then we take the basic object $(\widetilde{W''_{k+1}}^{\lambda}, (\mathfrak{a}''^{\lambda}_{k+1}, b''^{\lambda}), \widetilde{E''_{k+1}}^{\lambda})$ to be

$$\begin{cases} \widetilde{W''_{k+1}}^{\lambda} = \widetilde{W^{\lambda}_{k+1}} \\ {\mathfrak{a}''_{k+1}}^{\lambda} = (\mathfrak{a}^{\lambda}_{k+1})'' \text{ as constructed in Lemma 5-4} \\ b''^{\lambda} = (b^{\lambda})'' \text{ as constructed in Lemma 5-4} \\ \widetilde{E''_{k+1}}^{\lambda} = \widetilde{E^{\lambda}_{k+1}}^{+} = E^+_{k+1} \cap \widetilde{W''_{k+1}}^{\lambda}. \end{cases}$$

Let $\mathfrak{C}_{G,V}$ be the collection of all the sequences of transformations and smooth morphisms of pairs with specified closed subsets, starting with $(G_{k,V}, (V, {E''}_{k,V}))$, which satisfy condition (GB-1) with respect to the charts $\{(\widetilde{W''_{k+1}}^{\lambda}, (\mathfrak{a}''^{\lambda}_{k+1}, b''^{\lambda}), \widetilde{E''_{k+1}}^{\lambda})\}$.

Condition (GB-3) is trivially satisfied by the construction, whereas condition (GB-0) is a consequence of the statement of Lemma 5-4 for $N = 0$ and condition (GB-2) a consequence of the statement of Lemma 5-4 for N general. (Note that we shift the starting point for the lemmas to the stage $k + 1$.)

Therefore, the collection $\mathfrak{C}_{G,V}$ is represented by a general basic object $(\mathcal{G}_{k,V}, (V, E''{}_{k,V})$ over $(G_{k,V}, (V, E''{}_{k,V}))$ with a d-dimensional structure.

Now the "Moreover" part of Lemma 5-4 and the key inducive lemma (Lemma 3-1) imply that the general basic object $(\mathcal{G}_{k,V}, (V, E''{}_{k,V}))$, which represents the collection $\mathfrak{C}_{G,V}$, has a $(d-1)$-dimensional structure.

It also follows from Lemma 5-4 that the general basic object $(\mathcal{G}_{k,V}, (V, E''{}_{k,V}))$ has properties (α) and (β), which provide its characterization:

(α) With each sequence in $\mathfrak{C}_{G,V}$

$$(G_{k,V}, (V = V_k, E''{}_{k,V})) \overset{\pi''_{k+1,V}}{\longleftarrow} \cdots \overset{\pi''_{k+N,V}}{\longleftarrow} (G_{k+N,V}, (V_{k+N}, E''{}_{k+N,V}))$$

satisfying the condition

$$G_{k+j,V} \neq \emptyset \text{ for } j = 0, ..., N-1,$$

there corresponds an extension of the original sequence of transformations and smooth morphisms (where π_{k+1} is the open immersion $V \hookrightarrow W_k$)

$$\begin{aligned}&(F_0, (W_0, E_0)) \leftarrow \cdots \leftarrow (F_k, (W_k, E_k)) \leftarrow \\ &(F_{k+1}, (W_{k+1}, E_{k+1})) \overset{\pi_{k+2}}{\longleftarrow} \cdots \overset{\pi_{k+N+1}}{\longleftarrow} (F_{k+N+1}, (W_{k+N+1}, E_{k+N+1}))\end{aligned}$$

with condition

$$(\heartsuit') \quad \left\{\begin{array}{l} Y_{i-1} \subset \underline{\text{Max}}\ t_{i-1} \subset \underline{\text{Max}}\ w\text{-ord}_{i-1} (\subset F_{i-1}) \\ \text{whenever } \pi_i \text{ is a transformation with center } Y_{i-1} \end{array}\right\} \text{ for } i = k{+}2, ..., k{+}N{+}1$$

satisfying the following conditions:

(i) π''_{k+j+1} and π_{k+j+2} are the transformations with the same centers or the same smooth morphisms (as abstract varieties) for $j = 1, ..., N-1$ with $V_{k+j+1} = W_{k+j+2}$ (which means, in particular, if π''_{k+j+1} is the transformation with center $Y''_{k+j} \subset V_{k+j}$ which is permissible for $(G_{k+j,V}, (V_{k+j}, E''{}_{k+j,V}))$, then Y''_{k+j} is also permissible for $(F_{k+j+1}, (W_{k+j+1}, E_{k+j+1}))$,

(ii) we have

either
$$\left\{\begin{array}{l} \max\ t_{k+1} = \cdots = \max\ t_{k+N+1}, \text{ and} \\ G_{k+j,V} = \underline{\text{Max}}\ t_{k+j+1} \text{ for } j = 1, \cdots, N \end{array}\right.$$

or
$$\left\{\begin{array}{l} \max\ t_{k+1} = \cdots = \max\ t_{k+N} > \max\ t_{k+N+1} \\ (\text{or } \max\ t_{k+1} = \cdots = \max\ t_{k+N}\ \&\ F_{k+N+1} = \emptyset), \text{ and} \\ G_{k+j,V} = \underline{\text{Max}}\ t_{k+j+1} \text{ for } j = 0, ..., N-1\ \ \&\ \ G_{k+N,V} = \emptyset. \end{array}\right.$$

(β) Conversely, with each extension of the original sequence of transformations and smooth morphisms (where π_{k+1} is the open immersion $V \hookrightarrow W_k$)

$$
\begin{aligned}
&(F_0,(W_0,E_0)) \leftarrow \cdots \leftarrow (F_k,(W_k,E_k)) \leftarrow \\
&(F_{k+1},(W_{k+1},E_{k+1})) \overset{\pi_{k+2}}{\leftarrow} \cdots \overset{\pi_{k+N+1}}{\leftarrow} (F_{k+N+1},(W_{k+N+1},E_{k+N+1}))
\end{aligned}
$$

with condition

$$
(\heartsuit') \quad \left\{ \begin{array}{l} Y_{i-1} \subset \underline{\mathrm{Max}}\ t_{i-1} \subset \underline{\mathrm{Max}}\ w\text{-ord}_{i-1} (\subset F_{i-1}) \\ \text{whenever } \pi_i \text{ is a transformation with center } Y_{i-1} \end{array} \right\} \text{ for } i = k+2, ..., k+N+1
$$

and the condition

$$
\max\ t_{k+1} = \cdots = \max\ t_{k+N},
$$

there corresponds a sequence of transformations and smooth morphisms of general basic objects starting from $(G_{k,V},(V=V_k,E''_{k,V}))$

$$
(G_{k,V},(V=V_k,E''_{k,V})) \overset{\pi''_{k+1,V}}{\leftarrow} \cdots \overset{\pi''_{k+N,V}}{\leftarrow} (G_{k+N,V},(V_{k+N},E''_{k+N,V}))
$$

satisfying the condition

$$
G_{k+j,V} \neq \emptyset \text{ for } j = 0, ..., N-1
$$

and conditions (i) and (ii) as in (α).

Once we construct the general basic object $(\mathcal{G}_{k,V},(V,E''_{k,V}))$ with a $(d-1)$-dimensional structure, we have by induction the invariant ${f''}_k^{d-1} : G_{k,V} \to I_{d-1}$ attached to this general basic object (considered as a trivial sequence of general basic objects consisting only of itself).

Remark that the value ${f''}_k^{d-1}(p)$ is independent of the choice of the neighborhood V, since if we choose a different open neighborhood V', the general basic objects $(\mathcal{G}_{k,V},(V,E''_{k,V}))$ and $(\mathcal{G}_{k,V'},(V',E''_{k,V'}))$ restrict to the same general basic object $(\mathcal{G}_{k,V\cap V'},(V\cap V',E''_{k,V\cap V'}))$ (cf. Remark 4-2 (iv)(v)).

Note that we give the obvious lexicographical order (which is a total order) to the set
$I_d = \{\{0\} \times \Gamma^d\} \sqcup \{\mathbf{W}_{>0} \times \mathbf{T} \times \{\infty\}\} \sqcup \{\mathbf{W}_{>0} \times \mathbf{T} \times \{0\} \times I_{d-1}\}$ induced from the lexicographical orders on $\mathbf{W}, \Gamma^d, \mathbf{T}$ and I_{d-1}.

This completes the definition and construction of the invariant f^d.

Remark 9-2.

(i) In the definition above, we construct f^d from BOTTOM UP based upon the construction of f^{d-1} inductively. However, in reality, we can start writing down the invariant f^d from TOP DOWN without knowing what f^{d-1} would be: First we compute $w\text{-ord}_k(p)$. If $w\text{-ord}_k(p) = 0$, then go on to compute $\Gamma_k(p)$. If $w\text{-ord}_k(p) > 0$, then go on to compute $t_k(p)$. If $R(1)(q \in F_k; \{t_k(q) = t_k(p)\})_p \neq \emptyset$, then set the next factor to be ∞. If $R(1)(q \in F_k; \{t_k(q) = t_k(p)\})_p = \emptyset$, then set the next factor to be 0 and construct $(\mathcal{G}_{k,V},(V,E''_{k,V}))$. Now with the general

basic object $(\mathcal{G}_{k,V}, (V, E''_{k,V}))$ with a $(d-1)$-dimensional structue, we start writing down w-ord and repeat the same procedure as above, and so on.

(ii) Though in the definition of f^d above we used the charts $\{(\widetilde{W_k^\lambda}, (\mathfrak{a}_k^\lambda, b^\lambda), \widetilde{E_k^\lambda})\}$, the invariant f^d is completely determined only by the collection $\mathfrak{C}_i$ of the sequences of transformations and smooth morphisms with the specified closed subsets represented by the general basic object $(\mathcal{F}_i, (W_i, E_i))$ for $i = 0, \cdots, k$, the number d which refers to the dimension of the structure of the general basic objects and by the original sequence

$$(\mathcal{F}_0, (W_0, E_0)) \leftarrow \cdots \leftarrow (\mathcal{F}_k, (W_k, E_k)).$$

In fact, by Hironaka's trick, the invariants $w\text{-ord}_k$ and ord_k are determined by $\mathfrak{C}_k$ and by the number d referring to the dimension of the structure (cf. Definition-Proposition 4-5 and Remark 4-7). Therefore, Γ_k can also be determined by $\mathfrak{C}_k$, the number d, and the original sequence, as it can be computed purely by looking at ord_k, F_k and E_k. The invariant t_k is determined by $\mathfrak{C}_k$, the number d, and the original sequence also, as it can be computed purely by looking at $w\text{-ord}_k$ and E_k^-. Thus whether or not the codimension one part of the (local) maximum locus of the invariant t passes though a given point is also determined by $\mathfrak{C}_k$, the number d, and the original sequence. Now notice that the general basic object $(\mathcal{G}_{k,V}, (V, E''_{k,V}))$ is also determined by $\mathfrak{C}_k$, the number d, and the original sequence, as the collection $\mathfrak{C}_{G,V}$ is characterized by the loci $\underline{\text{Max}}\ t$ of the corresponding sequences in $\mathfrak{C}_k$ (sequel to the open immersion $(\mathcal{F}_k, (W_k, E_k)) \leftarrow (\mathcal{F}_{k+1}, (W_{k+1}, E_{k+1})))$. Therefore, ${f''}_k^{d-1}$ is also determined by $\mathfrak{C}_k$, the number d, and the original sequence.

(iii) The invariant f^d DOES depend on the number d specifying the dimension of the structure of your choice (of the general basic object) and is NOT purely determined by the collection $\mathfrak{C}_i$ of the sequences of transformations and smooth morphisms with the specified closed subsets, represented by the general basic object $(\mathcal{F}_i, (W_i, E_i))$ for $i = 0, \cdots, k$ and by the original sequence

$$(\mathcal{F}_0, (W_0, E_0)) \leftarrow \cdots \leftarrow (\mathcal{F}_k, (W_k, E_k)),$$

since so does the invariant w-ord (cf. Remark 4-7) and hence also the invariant t.

(iv) (Stability under smooth morphism) Let $\theta : (\mathcal{F}_0^\theta, (W_0^\theta, E_0^\theta)) \to (\mathcal{F}_0, (W_0, E_0))$ be a smooth morphism of general basic objects of relative dimension r, so that the dimension of the structure of $(\mathcal{F}_0^\theta, (W_0^\theta, E_0^\theta))$, induced by that of $(\mathcal{F}_0, (W_0, E_0))$ with a d-dimensional structure, is equal to $d + r$.

Let

$$(\mathcal{F}_0, (W_0, E_0)) \leftarrow \cdots \leftarrow (\mathcal{F}_k, (W_k, E_k))$$

be a sequence of transformations of general basic objects with a d-dimensional structure satisfying condition $(\heartsuit)$ and

$$(\mathcal{F}_0^\theta, (W_0^\theta, E_0^\theta)) \leftarrow \cdots \leftarrow (\mathcal{F}_k^\theta, (W_k^\theta, E_k^\theta))$$

be the sequence obtained by taking the Cartesian product of the first with the smooth morphism θ.

Then the second sequence is a sequence of transformations of general basic objects with a $(d+r)$-dimensional sructure satisfying condition ($\heartsuit$), and we have

$$f_k^{d+r}(p^\theta) = f_k^d(p) \text{ for any point } p \in F_k \text{ and } p^\theta \in F_k^\theta \cap \theta^{-1}(p).$$

(The verification is straightforward identifing the factors from TOP DOWN, and left to the reader as an exercise. (cf. Theorem 8-17)).

(v) (Stability under (analytic) localization) The invariant f^d is stable under (analytic) localization in the following sense: Let

$$(\mathcal{F}_0, (W_0, E_0)) \leftarrow \cdots \leftarrow (\mathcal{F}_k, (W_k, E_k))$$

be a sequence of transformations of general basic objects with d-dimensional structures satisfying condition ($\heartsuit$). Let $p = p_k \in F_k$ be a point and $p_i \in F_i$ its image in W_i.

Suppose we have another sequence of transformations of general basic objects with d-dimensional structures

$$(\mathcal{F}'_0, (W'_0, E'_0)) \leftarrow \cdots \leftarrow (\mathcal{F}'_k, (W'_k, E'_k))$$

with a point $p' = p'_k \in F'_k$ and its image $p'_i \in F'_i$ in W'_i.

Suppose we can find open neighborhoods V_i of p_i (resp. V'_i of p'_i) such that we have a commutative diagram of sequences of transformations of general basic objects, restricted to the open subsets, with vertical arrows being isomorphisms of general basic objects

$$\begin{array}{ccc}
(\mathcal{F}_0|_{V_0}, (V_0, E_0|_{V_0})) & \leftarrow \cdots \leftarrow & (\mathcal{F}_k|_{V_k}, (V_k, E_k|_{V_k})) \\
\downarrow & & \downarrow \\
(\mathcal{F}'_0|_{V'_0}, (V'_0, E'_0|_{V'_0})) & \leftarrow \cdots \leftarrow & (\mathcal{F}'_k|_{V'_k}, (V'_k, E'_k|_{V'_k})).
\end{array}$$

Or more generally suppose we have a commutative diagram of sequences of transformations of basic objects, restricted to the analytic neighborhoods, with vertical arrows being isomorphisms of (analytic) general basic objects

$$\begin{array}{ccc}
(\mathcal{F}_0, (W_0, E_0)) \times \mathrm{Spec}\ \widehat{\mathcal{O}_{W_0,p_0}} & \leftarrow \cdots \leftarrow & (\mathcal{F}_k, (W_k, E_k)) \times \mathrm{Spec}\ \widehat{\mathcal{O}_{W_k,p_k}} \\
\downarrow & & \downarrow \\
(\mathcal{F}'_0, (W'_0, E'_0)) \times \mathrm{Spec}\ \widehat{\mathcal{O}_{W'_0,p'_0}} & \leftarrow \cdots \leftarrow & (\mathcal{F}_k, (W_k, E_k)) \times \mathrm{Spec}\ \widehat{\mathcal{O}_{W'_k,p'_k}}.
\end{array}$$

Then we have

$$f_k^d(p) = f_k^d(p').$$

The verification is straightforward and left to the reader as an exercise.

(vi) (Invariance under action) Let

$$(\mathcal{F}_0, (W_0, E_0)) \leftarrow \cdots \leftarrow (\mathcal{F}_k, (W_k, E_k))$$

be a sequence of transformations of general basic objects with d-dimensional structures satisfying condition ($\heartsuit$). Let θ be an action on the general basic object $(\mathcal{F}_0, (W_0, E_0))$. Suppose the sequence is θ-equivariant. Then $f_k^d(p) = f_k^d(\theta(p))$ for any $p \in F_k$.

This is an immediate consequence of Lemma 8-10.

Theorem 9-3. *Let*

$$(\mathcal{F}_0, (W_0, E_0)) \leftarrow \cdots \leftarrow (\mathcal{F}_k, (W_k, E_k))$$

be the sequence of transformations of general basic objects with d-dimensional structures, obtained via the inductive algorithm (Theorem 5-1), representing resolution of singularities of a general basic object $(\mathcal{F}_0, (W_0, E_0))$.

Then the centers of the transformations are exactly the loci where the invariants f^d take their maxima, i.e.,

$$Y_{i-1} = \underline{\mathrm{Max}}\ f^d_{i-1} \textit{ for } i = 1, ..., k.$$

Proof.

This is a straightforward consequence of the description of the process of the inductive algorithm on how to choose the centers for the sequence representing resolution of singularities in Theorem 5-1 and the way we define the invariant f^d.

Note that the invariance of the invariant f^d and its stablity under smooth morphism (cf. Remark 9-2 (vi) and (iv)) give an alternative proof (or rather to say, a different presentation of the same proof), for the corresponding statements (cf. Proposition 8-11 Theorem 8-17) for the sequence of transformations representing resolution of singularities obtained via the inductive algorithm.

CHAPTER 10. NON-EMBEDDED RESOLUTION OF SINGULARITIES

In this chapter, we prove ((non-embedded) resolution of singularities), achieving Main Theme 0-1.

A variety X can be covered by a finite number of open subsets $\{U_s\}_{s \in S}$ which are embedded, as closed subschemes, into smooth varieties W_{U_s}, i.e., $U_s \subset W_{U_s}$. By choosing a number d sufficiently large and replacing W_{U_s} with $W_{U_s} \times \mathbb{A}^{d - \dim W_{U_s}}$, we may assume that all the ambient smooth varieties W_{U_s} are of the same dimension d.

We observe then, on the intersections $U_s \cap U_{s'}$ of the open subsets of the covering $\{U_s\}_{s \in S}$, not only that the invariants $f^d_{U_s}$ and $f^d_{U_{s'}}$, defined on the singular loci $U_s \subset W_{U_s}$ and $U_{s'} \subset W_{U_{s'}}$ of the basic objects $(W_{U_s}, (\mathcal{I}_{U_s}, 1), \emptyset)$ and $(W_{U_{s'}}, (\mathcal{I}_{U_{s'}}, 1), \emptyset)$ (and on their transformations) as in Chapter 9, coincide and hence give rise to a global invariant f^d_X on X (and its transformations), but also that the ideals defining the loci $\{f^d_{U_s} = \max f^d_X\} \subset W_{U_s}$ and $\{f^d_{U_{s'}} = \max f^d_X\} \subset W_{U_{s'}}$, restricted to $U_s \subset W_{U_s}$ and $U_{s'} \subset W_{U_{s'}}$, coincide and hence give rise to a global ideal of the center of blowup on X (and on its transformations).

Choosing the center(s) of blowup(s) this way, though based upon the method of embedded resolution of singularities, we obtain a sequence representing non-embedded resolution of singularities of X.

The sequence thus obtained is independent of the choice of the number d (though the invariant f^d_X is dependent of the number d) or the choice of the covering $\{U_s\}_{s \in S}$.

The sequence is equivariant with respect to any automorphism $\theta : X \xrightarrow{\sim} X$ in the sense that it lifts to an automorphism of the sequence.

Theorem 10-1 ((Non-embedded) Resolution of singularities). *Let X be a variety over a field k of characteristic zero. Take an open covering $\{U\}$ of X so that the open subsets U are embedded, as closed subschemes, into varieties W_U of dimension d smooth over k. (The number d is common to all the varieties W_U.)*

Then the invariants f^d_U, defined as in Chapter 9, on the singular loci $U \subset W_U$ of the basic objects $(W_U, (\mathcal{I}_U, 1), \emptyset)$ (as well as the invariants on their transformations) patch together to give rise to a global invariant f^d_X on X (as well as to the global invariants on the transformations of X). Moreover, the ideals, restricted to U, defining the loci $\{f^d_U = \max f^d_X\} \subset W_U$ (as well as the ideals, restricted to the transformations of U, defining the maximum loci of the invariants on the transformations of the original basic objects $(W_U, (\mathcal{I}_U, 1), \emptyset)$) patch together to determine the global ideal of the center of blowup on X (as well as the global ideals of the centers of blowups on the transformations of X).

This provides an algorithm to choose the centers of blowups, which lie over the singular locus $\mathrm{Sing}(X)$ of X, for constructing a sequence of (non-embedded) resolution of singularities of X.

The sequence thus obtained is independent of the choice of the number d (though the invariant f^d_X is dependent of the number d) or the choice of the covering $\{U\}$.

The sequence thus obtained is equivariant with respect to any automorphism $\theta : X \xrightarrow{\sim} X$ in the sense that it lifts to an automorphism of the sequence.

We give a more precise description of our algorithm in the following:

Inductively, we construct (a part of, i.e., up to the k-th stage of) the sequence representing (non-embedded) resolution of singularities

$$X = X_0 \overset{\pi_1}{\leftarrow} X_1 \overset{\pi_2}{\leftarrow} \cdots \overset{\pi_{k-1}}{\leftarrow} X_{k-1} \overset{\pi_k}{\leftarrow} X_k$$

with centers $Y_{i-1} \subset X_{i-1}$ for $i = 1, ..., k$ and the invariant $f^d_{X_i}$ on X_i for $i = 0, 1, ..., k$ with the following properties:

(i) For each closed point $p \in X$ and an open subset U containing p and taken from the open covering, there exists an open neighborhood $p \in U_p \subset U$ with an induced embedding $U_p \subset W_{U_p}$ such that

$$\psi_{i-1}(Y_{i-1}) \cap U_p \neq \emptyset \text{ if and only if } \psi^{-1}_{i-1}(p) \cap Y_{i-1} \neq \emptyset \text{ for } i = 1, ..., k$$

where $\psi_{i-1} = \pi_1 \circ \pi_2 \circ \cdots \circ \pi_{i-1}$.

(ii) The first $l_{p,k}$-stages of the sequence of transformations representing resolution of singularities of the basic object $(W_{U_p}, (\mathcal{I}_{U_p}, 1), \emptyset)$

$$(W_{U_p}, (\mathcal{I}_{U_p}, 1), \emptyset) = ((W_{U_p})_0, ((J_{U_p})_0, 1), (E_{U_p})_0) \leftarrow \cdots \leftarrow ((W_{U_p})_{l_{p,k}}, ((J_{U_p})_{l_{p,k}}, 1), (E_{U_p})_{l_{p,k}}),$$

where

$$l_{p,k} = \#\{i; \psi^{-1}_{i-1}(p) \cap Y_{i-1} \neq \emptyset, i = 1, ..., k\},$$

gives rise to a sequence representing (the first $l_{p,k}$-stages of) resolution of singularities of U_p

$$U_p = (U_p)_0 \leftarrow \cdots \leftarrow (U_p)_{l_{p,k}}$$

where $(U_p)_j$ is the strict transform of $U_p = (U_p)_0$ on $(W_{U_p})_j$ for $j = 1, ..., l_{p,k}$.

This sequence coincides with

$$X = X_0 \overset{\pi_1}{\leftarrow} X_1 \overset{\pi_2}{\leftarrow} \cdots \overset{\pi_{k-1}}{\leftarrow} X_{k-1} \overset{\pi_k}{\leftarrow} X_k$$

restricted over U_p where we ignore the trivial transformations π_i whenever $\psi^{-1}_{i-1}(U_p) \cap Y_{i-1} = \emptyset$.

(We remark that, in the above sequence of basic objects starting from $(W_{U_p}, (\mathcal{I}_{U_p}, 1), \emptyset)$, we only consider the neighborhoods of the strict transforms $(U_p)_j$ of $(U_p)_0 = U_p$. That is to say, in the sequence representing resolution of singularities of $(W_{U_p}, (\mathcal{I}_{U_p}, 1), \emptyset)$, we ignore the transformations whose centers of blowups are away from the strict transforms.)

(iii) The invariant $f^d_{X_k}(p_k)$ for $p_k \in \psi_k^{-1}(U_p) \subset X_k$, where $\psi_k^{-1}(U_p) = (U_p)_{l_{p,k}}$ with $\psi_k = \pi_1 \circ \cdots \circ \pi_k$, is defined to be equal to the value $f^d_{U_p, l_{p,k}}(p_k)$, where the invariant $f^d_{U_p, l_{p,k}}$ is attached to the $l_{p,k}$-th stage of the sequence of basic objects given in (ii) and where the invariant $f^d_{U_p, l_{p,k}}$ is defined on the singular locus of the basic object $((W_{U_p})_{l_{p,k}}, ((J_{U_p})_{l_{p,k}}, 1), (E_{U_p})_{l_{p,k}})$, which contains $(U_p)_{l_{p,k}}$.

The invariant $f^d_{U_p, l_{p,k}}(p_k)$ is independent of the choice of U or U_p (justifying the omission of reference to U_p or U in the notation $f^d_{X_k}$).

(Note that the invariant f^d is stable (invariant) under open immersion. Thus the invariant $f^d_{U_p}$ is just the restriction of f^d_U over U_p.)

(iv) The defining ideal $\mathcal{I}_{Y_k}$ of the center $Y_k \subset X_k$, which may not be smooth, reduced or irreducible in general, is taken so that $\mathcal{I}_{Y_k}|_{\psi_k^{-1}(U_p)}$ coincides with the ideal, restricted to $(U_p)_{l_{p,k}} = \psi_k^{-1}(U_p)$, defining the locus $\{f^d_{U_p, l_{p,k}} = \max f^d_{X_k}\}$ inside of $(W_{U_p})_{l_{p,k}}$.

(v) The center Y_k lies over the singular locus $\mathrm{Sing}(X)$ of X.

Remark 10-2.

(i) As we stated in the footnote to Main Theme 0-1, we do not require that the centers $Y_i \subset X_i \ \ (i = 0, ..., k)$ to be smooth or reduced (and we also allow the centers to be reducible). In fact, our algorithm produces centers which may not be smooth or reduced. Therefore, though it is true that set-theoretically we have

$$\mathrm{Supp}(Y_i) = \underline{\mathrm{Max}}\ f^d_{X_i} = \{p \in X_i; f^d_{X_i}(p) = \max f^d_{X_i}\},$$

this description of the center is not enough to determine its scheme-theoretic structure. This feature is in clear contrast to the situiation where, if we required the centers to be (smooth and) reduced, the set-theoretic description of them as the maximum loci of the invariants would suffice.

(ii) We remark that it is NOT sufficient merely to prove, in order to construct a global sequence of non-embedded resolution of singularities, that X has an open covering $\{U\}$ with embeddings $U \subset W_U$ into varieties smooth over k and that the process of embedded resolution of $U \subset W_U$ restricted to $U \cap V$ "coincides in the naive sense" with the process of embedded resolution of $V \subset W_V$ restricted to $U \cap V$ for any two open subsets U and V of the open covering. The reason, which involves the interpretation of the words "coincides in the naive sense", is the following:

When we restrict the process to a smaller open subset $U \cap V$, we ignore the trivial transformations blowing up the centers outside of $U \cap V$. So even if we prove the processes, obtained by restricting those over U and V, coincide after ignoring those trivial transformations, we would be at loss, without introducing an invariant, about how to patch the processes for U and V together including the transformations with centers outside of $U \cap V$, in order to obtain a global order of choosing centers.

Proof.

Let $\overline{k}$ be the algebraic closure of the field k. If we prove the assertion over $\overline{k}$, i.e., for $X \times \mathrm{Spec}\ \overline{k}$ with its open covering $\{U \times \mathrm{Spec}\ \overline{k}\}$ and embeddings $U \times \mathrm{Spec}\ \overline{k} \hookrightarrow W_U \times \mathrm{Spec}\ \overline{k}$ (We may lose the irreducibility assumption on "varieties", but since the theory remains valid without any change in the argument, we ignore this point.), together with the equivariance assertion (which obviously implies the equivariance under the action of the Galois group $\mathrm{Gal}(\overline{k}/k)$), then the assertion over k follows. Therefore, we may assume that k is algebraically closed in what follows.

We check the inductive construction of our algorithm (stated in "a more precise description" in the statement of Theorem 10-1), condition by condition, starting with the $(k = 0)$-th stage.

$\boxed{\text{Case } k=0}$

We look at the case when $k=0$.

Condition (i) is obvious at the stage $k=0$, choosing $p \in U_p \subset U$ where U is an open subset containg p and taken from the covering and where U_p is any open neighborhood $p \in U_p \subset U$

Condition (ii) is clear at the stage $k=0$.

We verify condition (iii) in the following.

Let $e = \dim m_{X,p}/m_{X,p}^2$ be the embedding dimension of X at p.

Take a small open neighborhood U_p so that we may assume that it is embedded into a smooth affine variety $W_{U_p} = \text{Spec } A(W_{U_p})$ of dimension d with affine coordinate ring $A(W_{U_p})$ and that there exists a system of regular parameters $(x_1, ..., x_{d-e}, y_1, ..., y_e)$ satisfying the following properties:

(α) $\langle x_1, ..., x_{d-e}\rangle \subset \Gamma(W_{U_p}, \mathcal{I}_{U_p}) = I_{U_p}$, and
(β) we have a commutative diagram

$$\begin{array}{ccccccc}
0 & \to & \mathcal{K} & \to & k[[Y_1, ..., Y_e]] & \to & \widehat{\mathcal{O}_{X,p}} \to 0 \\
\| & & \uparrow & & \uparrow & & \| \\
0 & \to & I_{U_p}/\langle x_1, ..., x_{d-e}\rangle \otimes \widehat{\mathcal{O}_{W_{U_p},p}} & \to & A(W_{U_p})/\langle x_1, ..., x_{d-e}\rangle \otimes \widehat{\mathcal{O}_{W_{U_p},p}} & \to & \widehat{\mathcal{O}_{U_p,p}} \to 0
\end{array}$$

where the second middle vertical arrow is an isomorphism sending $y_1, ..., y_e$ to $Y_1, ..., Y_e$, regarded as independent variables.

This induces an isomorphism of (analytic) basic objects

$$\begin{aligned}
&(W_{U_p}, (\mathcal{I}_{U_p}, 1), \emptyset)/(x_1, ..., x_{d-e}) \times \text{Spec } \widehat{\mathcal{O}_{W,p}} \\
&\qquad \xrightarrow{\sim} ((\widehat{\mathbb{A}^e})_0 = \text{Spec } k[[Y_1, ..., Y_e]], (\mathcal{K}_0 = \mathcal{K}, 1), (\mathcal{E}_p)_0 = \emptyset),
\end{aligned}$$

where "$/\langle x_1, ..., x_{d-e}\rangle$" denotes the restriction to the nonsingular subvariety defined by the ideal $\langle x_1, ..., x_{d-e}\rangle$, by abuse of notation.

We now compute the invariant $f^d_{U_p,0}(p)$.

<u>Subcase $e - \dim X > 0$</u>: We consider the subcase $e - \dim X > 0$.

Suppose $d - e > 0$.

First, since $x_1 \in (J_{U_p})_0 = \overline{(J_{U_p})_0} = \mathcal{I}_{U_p}$, we have

$$w\text{-ord}^d_{U_p,0}(p) = 1 \text{ and } t^d_{U_p,0}(p) = (1, 0).$$

Since $d - e + e - \dim X > 1$, the next factor is 0 (cf. Chapter 9). Thus we have the pattern

$$(1, (1, 0), 0).$$

Now we look at the general basic object with a $(d-1)$-dimensional structure as in Definition-Construction 9-1, whose chart at p is given by $(W_{U_p}, (\mathcal{I}_{U_p}, 1), \emptyset)/\langle x_1\rangle$ by construction (cf. Lemma 5-3 and Lemma 5-4). Note that, since $b = b_0 =$

$b' = b'' = 1$, the coefficient ideal is nothing but the restriction of the original ideal $(J_{U_p})_0 = \overline{(J_{U_p})_0} = \mathcal{I}_{U_p}$ to the smooth hypersurface $\{x_1 = 0\}$.

Suppose $d - e - 1 > 0$.

First, since $x_2 \in (J_{U_p}/\langle x_1 \rangle)_0 = \overline{(J_{U_p}/\langle x_1 \rangle)_0} = \mathcal{I}_{U_p}/\langle x_1 \rangle$, we have

$$w\text{-ord}^{d-1}_{U_p/\langle x_1 \rangle, 0}(p) = 1 \text{ and } t^{d-1}_{U_p/\langle x_1 \rangle, 0}(p) = (1, 0).$$

Since $d - e - 1 + e - \dim X > 1$, the next factor is 0. Thus we have the pattern

$$(1, (1, 0), 0, 1, (1, 0), 0).$$

Now we look at the general basic object with a $(d-2)$-dimensional structure as in Definition-Construction 9-1, whose chart at p is given by $(W_{U_p}, (\mathcal{I}_{U_p}, 1), \emptyset)/\langle x_1, x_2 \rangle$ by construction (cf. Lemma 5-3 and Lemma 5-4). Note that, since $b = b_0 = b' = b'' = 1$, the coefficient ideal is nothing but the restriction of the ideal $(J_{U_p}/\langle x_1 \rangle)_0 = \overline{(J_{U_p}/\langle x_1 \rangle)_0} = \mathcal{I}_{U_p}/\langle x_1 \rangle$ to the smooth subvariety $\{x_1 = x_2 = 0\}$.

Inductively, carrying out the same argument with $x_3, ..., x_{d-e}$, we conclude that after repeating the pattern $(1, (1, 0), 0)$ for $(d-e)$-times

$$(1, (1, 0), 0, 1, (1, 0), 0, \cdots, 1, (1, 0), 0)$$

we reach the general basic object with a $(e = d - (d - e))$-dimensional structure, whose chart at p is given by $(W_{U_p}, (\mathcal{I}_{U_p}, 1), \emptyset)/\langle x_1, \cdots, x_{d-e} \rangle$, which is analytically isomorphic to $((\widehat{\mathbb{A}^e})_0, (\mathcal{K}_0, 1), (\mathcal{E}_p)_0)$.

It is straightforward to see that the analytic basic object $((\widehat{\mathbb{A}^e})_0, (\mathcal{K}_0, 1), (\mathcal{E}_p)_0)$ is purely determined by $\widehat{\mathcal{O}_{X,p}}$, and independent of the choice of U, $U_p \subset U$, or a system of regular parameters $(x_1, ..., x_{d-e}, y_1, ..., y_e)$. In fact, let $p \in U'_p \subset U'$ be another choice of open subsets with a system regular parameters $(x'_1, ..., x'_{d-e}, y'_1, ..., y'_e)$ as above, which leads to a commutative diagram

$$\begin{array}{ccccccccc}
0 & \rightarrow & \mathcal{K}' & \rightarrow & k[[Y'_1, ..., Y'_e]] & \rightarrow & \widehat{\mathcal{O}_{X,p}} & \rightarrow & 0 \\
\| & & \uparrow & & \uparrow & & \| & & \\
0 & \rightarrow & I_{U_p}/\langle x'_1, ..., x'_{d-e} \rangle \otimes \widehat{\mathcal{O}_{W_{U_p},p}} & \rightarrow & A(W_{U_p})/\langle x'_1, ..., x'_{d-e} \rangle \otimes \widehat{\mathcal{O}_{W_{U_p},p}} & \rightarrow & \widehat{\mathcal{O}_{U_p,p}} & \rightarrow & 0.
\end{array}$$

Then there exists an isomorphism (though non-canonical)

$$\phi : k[[Y_1, ..., Y_e]] \rightarrow k[[Y'_1, ..., Y'_e]]$$

which makes the following diagram commute

$$\begin{array}{ccccccccc}
0 & \rightarrow & \mathcal{K} & \rightarrow & k[[Y_1, ..., Y_e]] & \rightarrow & \widehat{\mathcal{O}_{X,p}} & \rightarrow & 0 \\
\| & & \downarrow & & \downarrow \phi & & \| & & \\
0 & \rightarrow & \mathcal{K}' & \rightarrow & k[[Y'_1, ..., Y'_e]] & \rightarrow & \widehat{\mathcal{O}_{X,p}} & \rightarrow & 0.
\end{array}$$

Thus denoting the invariant attached to the analytic basic object $((\widehat{\mathbb{A}^e})_0, (\mathcal{K}_0, 1), (\mathcal{E}_p)_0)$ by $\widehat{f^e_{p,0}}$, we conclude

$$f^d_{X_0}(p) = f^d_{U_p,0}(p) = (1,(1,0),0,1,(1,0),0,\cdots,1,(1,0),0,\widehat{f^e_{p,0}}(p)),$$

where the pattern $(1,(1,0),0)$ is repeated for $(d-e)$-times. This is independent of the choice of U, $U_p \subset U$, or a system of regular parameters $(x_1,...,x_{d-e},y_1,...,y_e)$.

Suppose $d-e=0$. Then we conclude

$$f^d_{X_0}(p) = f^d_{U_p,0}(p) = \widehat{f^e_{p,0}}(p).$$

<u>Subcase $e-\dim X = 0$</u>: We consider the remaining subcase $e-\dim X = 0$.

In the case $e-\dim X=0$, we conclude by a similar consideration

$$f^d_{X_0}(p) = f^d_{U_p,0}(p) = (1,(1,0),0,1,(1,0),0,\cdots,1,(1,0),0,1,(1,0),\infty),$$

where the pattern $(1,(1,0),0)$ is repeated for $(d-e-1)$-times. This is also independent of the choice of U, $U_p \subset U$, or a system of regular parameters $(x_1,...,x_{d-e},y_1,...,y_e)$. Observe that in this case, i.e., when p is a nonsingular point of X, the value $f^d_{X_0}(p)$ is minimum, i.e.,

$$f^d_{X_0}(p) \leq f^d_{X_0}(q) \quad \forall q \in X.$$

Now we look at condition (iv), using the same notation as for the verification of condition (iii).

The locus $\{q \in X_0; f^d_{U_p,0}(q) = \max\ f^d_X\} \subset W_{U_p} = (W_{U_p})_0$ can be identified, after taking the product $\times \mathrm{Spec}\ \widehat{\mathcal{O}_{W,p}}$, with the locus $M_{0,p} = \{q \in (\widehat{\mathbb{A}^e})_0; (1,(1,0),0,\cdots,1,(1,0),0,\widehat{f^e_{p,0}}(q)) = \max f^d_X\}$ where the pattern $(1,(1,0),0)$ is repeated for $(d-e)$-times, via the inclusion $(\widehat{\mathbb{A}^e})_0 \subset \mathrm{Spec}\ \widehat{\mathcal{O}_{W,p}}$, i.e.,

$$\begin{array}{ccc} (\widehat{\mathbb{A}^e})_0 & \hookrightarrow & \mathrm{Spec}\ \widehat{\mathcal{O}_{W,p}} \\ \cup & & \cup \\ M_{0,p} & = & \{q \in X_0; f_{(U_p)_0}(q) = \max f^d_X\} \times \mathrm{Spec}\ \widehat{\mathcal{O}_{W,p}}. \end{array}$$

Therefore, the defining ideal $\mathcal{I}_{Y_0}$ of the center $Y_0 \subset X_0$, is characterized analytically locally as the ideal defining the locus $M_{0,p} = \{q \in (\widehat{\mathbb{A}^e})_0; (1,(1,0),0,\cdots,1,(1,0),0,\widehat{f^e_{p,0}}(q)) = \max f^d_X\}$, restricted to $\mathrm{Spec}\ \widehat{\mathcal{O}_{X,p}} \subset (\widehat{\mathbb{A}^e})_0$. As before, it is straightforward to see that this characterization is independent of the choice of U_p, $U_p \subset U$, or a system of regular parameters $(x_1,...,x_{d-e},y_1,...,y_e)$, in the following sense: For any other choice of open subsets or a system of regular parameters, there exists an isomorphism ϕ, as in the verification of condition (iii), which identifies the ideals defining $\mathrm{Spec}\ \widehat{\mathcal{O}_{X,p}}$ in $\mathrm{Spec}\ \widehat{\mathcal{O}_{W,p}}$ (i.e., $\mathcal{K}$ and $\mathcal{K}'$) as closed subschemes, as well as the ideals defining the centers. Therefore, via ϕ, we identify the two a priori different defining ideals of the centers, restricted to $\mathrm{Spec}\ \widehat{\mathcal{O}_{X,p}}$, as one.

This finishes the verification of condition (iv) at the $(k = 0)$-th stage.

Condition (v) follows from the observation in Subcase $e - \dim X = 0$ that $f^d_{X_0}(p)$ is minimum when p is a nonsingular point and hence that when the center contains p the entire X has to be an irreducible component of the center, which implies X is already nonsingular.

This completes checking of the conditions at the $(k = 0)$-th stage.

$\boxed{\text{Inductional Assumption of Case } k = k}$

Before proving the assertions for the case $k = k + 1$, based upon the assertions for the case $k = k$, we make some extra inductional assumption explicit for the case $k = k$ aside from condition (i) through (iv).

Subcase $e - \dim X > 0$: In fact, we assume inductively, starting with

$$\begin{aligned}(W_{U_p}, (\mathcal{I}_{U_p}, 1), \emptyset)/(x_1, ..., x_{d-e}) &\times \text{Spec } \widehat{\mathcal{O}_{W,p}} \\ &= ((W_{U_p})_0, ((J_{U_p})_0, 1), (E_{U_p})_0)/(x_{1,0}, ..., x_{d-e,0}) \times \text{Spec } \widehat{\mathcal{O}_{W,p}} \\ (\widehat{\mathbb{A}^e}, (\mathcal{K}, 1), \emptyset) &= ((\widehat{\mathbb{A}^e})_0, (\mathcal{K}_0, 1), (\mathcal{E}_p)_0),\end{aligned}$$

we have a commutative diagram between the sequences of transformations of (analytic) basic objects

$$\begin{array}{ccc} ((W_{U_p})_0, ((J_{U_p})_0, 1), (E_{U_p})_0)/(x_{1,0}, ..., x_{d-e,0}) \times \text{Spec } \widehat{\mathcal{O}_{W,p}} & \overset{\sim}{\rightarrow} & ((\widehat{\mathbb{A}^e})_0, (\mathcal{K}_0, 1), (\mathcal{E}_p)_0) \\ \uparrow & & \uparrow \\ \cdot & & \cdot \\ \cdot & & \cdot \\ \cdot & & \cdot \\ \uparrow & & \uparrow \\ ((W_{U_p})_{l_{p,k}}, ((J_{U_p})_{l_{p,k}}, 1), (E_{U_p})_{l_{p,k}})/(x_{1,l_{p,k}}, ..., x_{d-e,l_{p,k}}) \times \text{Spec } \widehat{\mathcal{O}_{W,p}} & \overset{\sim}{\rightarrow} & ((\widehat{\mathbb{A}^e})_{l_{p.k}}, (\mathcal{K}_{l_{p,k}}, 1), (\mathcal{E}_p)_{l_{p,k}}) \end{array}$$

where the $x_{j,i}$ denote the strict transform of $x_{j,0} = x_j$, i.e., the equations defining the strict transforms of the smooth hypersurfaces $\{x_j = 0\}$.

(Remark that only at this place of the notes we write down the sequences of transformations vertically, due to the limitation of space.)

Note that this commutative diagram leads us immediately to the computation of the invariant $f^d_{X_k}(p_k)$ for $p_k \in \psi_k^{-1}(p)$ as was done in the case $k = 0$

$$f^d_{X_k}(p_k) = f^d_{U_p, l_{p,k}}(p_k) = (1, (1, 0), 0, \cdots, 1, (1, 0), 0, \widehat{f^d_{p,l_{p,k}}}(p_k)),$$

where the pattern $(1, (1, 0), 0)$ is repeated for $(d - e)$-times and where $\widehat{f^d_{p,l_{p,k}}}$ denotes the invariant attached to the (analytic) basic object $((\widehat{\mathbb{A}^e})_{l_{p.k}}, (\mathcal{K}_{l_{p,k}}, 1), (\mathcal{E}_p)_{l_{p,k}})$ obtained from the sequence above.

<u>Subcase $e - \dim X = 0$</u>: Note that in this case X is smooth in a neighborhood of p and we have not touched this neighborhood in the process of (non-embedded) resolution of singularities. Just as at the 0-th stage, we have

$$f^d_{X_k}(p_k) = f^d_{U_p, l_{p,k}}(p_k) = (1, (1, 0), 0, \cdots, 1, (1, 0), 0, 1, (1, 0), \infty),$$

where the pattern $(1, (1, 0), 0)$ is repeated for $(d - e - 1)$-times. Observe that in this case the value $f^d_{X_k}(p)$ is minimum, i.e.,

$$f^d_{X_k}(p) \leq f^d_{X_k}(q) \quad \forall q \in X_k.$$

$\boxed{\text{Case } k = k + 1}$

We look at the assertions for the case $k = k + 1$.

If $\psi_k^{-1}(p_k) \cap Y_k \neq \emptyset$, then we choose $p \in U_p \subset U$ just as we did at the k-th stage. If $\psi_k^{-1}(p_k) \cap Y_k = \emptyset$, then we shrink U_p so that $U_p \cap \psi_k(Y_k) = \emptyset$. This shrinking can be done, since $\psi_k(Y_k)$ is closed by properness of ψ_k and since $p_k \notin \psi_k(Y_k)$. For this choice of $p \in U_p \subset U$, condition (i) is satisfied at the stage $k = k + 1$.

Condition (ii) immediately follows from the construction, since in case $\psi_k^{-1}(p_k) \cap Y_k \neq \emptyset$, the center Y_k over $(U_p)_{l_{p,k}}$ is given as the restriction to $(U_p)_{l_{p,k}}$ of the center of the transformation of basic objects

$$((W_{U_p})_{l_{p,k}}, ((J_{U_p})_{l_{p,k}}, 1), (E_{U_p})_{l_{p,k}}) \leftarrow ((W_{U_p})_{l_{p,k+1}}, ((J_{U_p})_{l_{p,k+1}}, 1), (E_{U_p})_{l_{p,k+1}}).$$

Note that in case $\psi_k^{-1}(p_k) \cap Y_k = \emptyset$ we have $l_{p,k} = l_{p,k+1}$.

In condition (iii), the only thing to check is that the invariant $f^d_{U_p, l_{p,k+1}}(p_{k+1})$ is independent of the choice of U or U_p. This can be seen easily, once one realizes that the vertical commutative diagram in the inductional assumption of the case $k = k$ can be extended to the case $k = k + 1$, giving an anlytic characterization of the invariant, and that for another choice $p \in U'_p \subset U'$ the isomorphism ϕ (discussed in the verification of the assertions at the stage $k = 0$) can be extended to an isomorphism between the two sequences of analytic basic objects all the way to the stage $k = k + 1$.

Condition (iv) indicates how to choose the next center $Y_{k+1} \subset X_{k+1}$ of blowup, which follows from Theorem 9-3 and from the analytic characterization of the ideal defining the center. This analytic characterization can be verified via an argument identical to the one for the verification of condition (iii).

Condition (v) follows from the observation in Subcase $e - \dim X = 0$ that $f^d_{X_k}(p)$ is minimum when p is a nonsingular point of X and hence that when the center contains p the entire strict transform has to an irreducible component of the center, which implies X_k is already nonsingular.

This finishes checking the inductive construction of our algorithm.

The sequence obtained through our algorithm is independent of the covering $\{U\}$, as it is purely determined by the analytic basic objects which are independent of the covering.

If we choose a number d' (say $d' > d$) for the common dimension of the smooth ambient varieties W_U, then the invariant $f^{d'}_{X_k}$ only differs from $f^{d}_{X_k}$ by repeating the pattern $(1,(1,0),0)$ for $(d'-d)$-times. However, the analytic basic objects $(\widehat{\mathbb{A}},(\mathcal{K},1),(\mathcal{E}_p)_0)$ (and their transformations), being of the same dimension as the embedding dimension, remain unchanged. These analytic objects are the only ingredients to determine the centers of the transformations of the sequence. Therefore, the sequence obtained through our algorithm is independent of the number d.

Finally, we can see that any automorphism $\theta : X \overset{\sim}{\to} X$ can be lifted to an automorphism of the sequence, once we realize that $\{\theta(U)\}$ gives another open covering of X and then that the argument as above showing the independence of the sequence of the choice of the covering also shows the lifting of the automorphism.

This completes the proof for Theorem 10-1.

CHAPTER 11. EXAMPLES

In this chapter, we present examples, some of which demonstrate a couple of essential points of the inductive algorithm and some of which simply demonstrate how it works. Many of them are communicated to the author by Profs. Encinas and Villamayor, and/or taken directly from the lectures delivered by the latter at Purdue University. (However, any inaccuracy in the presentation is solely the responsibility of the author.)

Example 11-1 (Why do we need to keep the history ?).

In our inductive algorithm of resolution of singularities of a (general) basic object, the t-invariant plays a key role. The second factor n_k of the t-invariant $t_k = (w\text{-ord}_k, n_k)$ depends upon the "history" of the process. Namely we have to look at the sequence from the 0-th stage up to the k-th stage, finding when the maximum of the invariant w-ord changed in the past (cf. Definition 1-10).

Do we really have to keep track of the history of the process of resolution of singularities or principalization ?

The following simple example shows that the answer is *yes* , not only in our inductive algorithm but also in *any* algorithm (which looks only at the weak transforms of the ideal), in the following sense:

Suppose we look for an algorithm of principalization, which assigns a uniquely determined sequence to a gievn ideal $\mathcal{I} \subset \mathcal{O}_W$ on a smooth variety, satisfying conditions (i), (ii), and (iii) in Main-Theme 0-3, and the following extra requirements:
(α) the algorithm is equivariant with respect to an action, and
(β) the algorithm is stable with respect to truncation and localization.

The example below shows that there is NO such algorithm.

In other words, condition (β) inevitably leads to an infinite loop in the algorithm. Hence, in order to guarantee that an algorithm of principalization come to an end after finitely many steps, we have to give up condition (β).

Note that the precise meaning of condition (β) is:

Let
$$W_0 \overset{\pi_1}{\leftarrow} W_1 \overset{\pi_2}{\leftarrow} \cdots \overset{\pi_{l-1}}{\leftarrow} W_{l-1} \overset{\pi_l}{\leftarrow} W_l$$
and
$$W'_0 \overset{\pi_1}{\leftarrow} W'_1 \overset{\pi_2}{\leftarrow} \cdots \overset{\pi_{l'-1}}{\leftarrow} W'_{l'-1} \overset{\pi_{l'}}{\leftarrow} W'_{l'}$$
be sequences constructed according to the algorithm for principalization of ideals $\mathcal{I} = \mathcal{I}_0 = \overline{\mathcal{I}_0} \subset \mathcal{O}_W$ on a smooth variety $W = W_0$ and $\mathcal{I}' = \mathcal{I}'_0 = \overline{\mathcal{I}'_0} \subset \mathcal{O}_{W'}$ on $W' = W'_0$. Let $\overline{\mathcal{I}_l}$ and $\overline{\mathcal{I}'_{l'}}$ be the weak transforms of $\mathcal{I}$ and $\mathcal{I}'$, respectively (cf. Remark 1-11 (ii)).

Suppose there exist open subsets $U \subset W_l$ and $V \subset W'_{l'}$ such that there is an isomorphism $U \overset{\sim}{\to} V$ which induces an isomorphism of ideals $\overline{\mathcal{I}_l}|_U \overset{\sim}{\to} \overline{\mathcal{I}'_{l'}}|_V$.

Then the extensions of the sequences of principalization of ideal, constructed according to the algorithm, coincide over U and over V (after ignoring the trivial transformations whose centers lie outside of the loci over U or V).

We translate the truncation property as looking only at the present (situation of the weak transform). Therefore, we interpret the necessity to give up condition (β) as the need to look into the history.

Let

$$\mathcal{I} = \langle x_1, x_2x_3 \rangle \subset \mathcal{O}_W \text{ where } W = \text{Spec } k[x_1, x_2, x_3].$$

Since J is not principal, we have to choose a center $Y \subset W$. By condition (i) of Main Theme 0-3, we have $Y \subset \text{Supp } \mathcal{O}_W/\mathcal{I}$. By condition (ii) of Main Theme 0-3, the center Y has to be smooth. There is an obvious action of $\mathbb{Z}_2$ on W, switiching x_2 and x_3, under which $\mathcal{I}$ is invariant. Therefore, Y has to be invariant under the $\mathbb{Z}_2$-action.

It is easy to see that the only center which satisfy all the above requirements and which includes the origin is the origin itself.

Therefore, the first blowup (in a neighborhood of the origin) of principalization *must* be along the ideal $\langle x_1, x_2, x_3 \rangle$.

However, over the open subset U of W_1 with the system of regular parameters (t_1, x_2, t_3) with

$$x_1 = t_1x_2, x_2 = x_2, x_3 = t_3x_2,$$

the weak transform $\overline{\mathcal{I}_1}|_U$ is in the identical form to the ideal $\mathcal{I}_0 = \overline{\mathcal{I}_0}$, i.e.,

$$\overline{\mathcal{I}_1}|_U = \langle t_1, x_2t_3 \rangle \xrightarrow{\sim} \overline{\mathcal{I}_0}|_V$$

where we set $V = W = W_0$.

Now it is clear that condition (β) leads to an infinite loop of the process induced by the algorithm.

Example 11-2 (Fundamental obstruction to carry out our algorithm in positive characteristic).

The success of our inductive algorithm of resolution of singularities depends in an essential way on finding a hypersurafce of maximal contact (cf. Remark 1-5, Lemma 3-1 (key inductive lemma)).

The following example shows that a hypersurface of maximal contact does *not* always exist in positive characteristic, and hence that there is a fundamental obstruction to carry out our algorithm in positive characteristic. (It is brought to the attention of the author by Prof. J. Włodarczyk via communication with Prof. P. Milman. The reader is also encouraged to look at Moh [1].)

Consider a hypersurafec singularity

$$0 \in \{f = 0\} \subset \mathbb{A}^4 = \text{Spec } k[x_1, x_2, x_3, x_4]$$

where

$$f = x_4^2 + x_1^3x_2 + x_2^3x_3 + x_3^7x_1.$$

Suppose that the characteristic of the base field k (which is assumed to be algebraically closed for simplicity) is equal to 2, i.e.,

$$\text{char}(k) = 2.$$

We look for the locus where the multiplicity of f is equal to 2 (or more), where 2 is the multiplicity of f at the origin.

By substituting the following into f

$$\begin{cases} x_1 = y_1 + a \\ x_2 = y_2 + b \\ x_3 = y_3 + c \\ x_4 = y_4 + d, \end{cases}$$

we see

$$f = \{d^2 + a^3 b + b^3 c + c^7 a\} + \{(a^2 b + c^7) y_1 + (b^2 c + a^3) y_2 + (ac^6 + b^3) y_3\} \\ + \text{ higher terms.}$$

It is straightforward to see from this that the locus of multiplicity 2 (or more) has the parametrization

$$\begin{cases} a = t^{15} \\ b = t^{19} \\ c = t^7 \\ d = t^{32}. \end{cases}$$

The embedding dimension of the curve parametrized as above at the origin is 4, and hence it can never be contained in a smooth hypersurface in a neighborhood of the origin. Therefore, there is no hypersurface of maximal contact at the origin for this example.

Example 11-3 (Our algorithm of resolution of singularities of a general basic object DOES depend on the specification of the dimension d of its structure.).

Our inductive algorithm of resolution of singularities of a general basic object $(\mathcal{F}, (W, E))$ is determined by the invariant f^d (cf. Chapter 9) where d is the number specifying the dimension of its structure. Sometimes the general basic object can have a d-dimensional structure as well as a d'-dimensional structure for two different numbers $d \neq d'$. That is to say, we can have two different sets of charts $\{(\widetilde{W^\lambda}, (\mathfrak{a}^\lambda, b^\lambda), \widetilde{E^\lambda})\}_{\lambda \in \Lambda}$ and $\{(\widetilde{W^\mu}, (\mathfrak{b}, c^\mu), \widetilde{E^\mu})\}_{\mu \in M}$ being of different dimensions d and d', i.e., $\dim \widetilde{W^\lambda} = d \neq d' = \widetilde{W^\mu}$, but giving rise to the same collection $\mathfrak{C}$ of sequences of smooth morphisms and transformations of pairs with specified closed subsets, represented by the general basic object $(\mathcal{F}, (W, E))$.

Since the invariants f^d and $f^{d'}$ could be different (cf. Remark 4-7 and Remark 9-2 (iii)), it is natural to suspect that our algorithm depends on the specification of the dimension of the structure of one's choice of a general basic object.

The following example, taken directly from Encinas [1], shows that this is indeed the case, demonstrating a general basic object having two different sequences of resolution of singularities, though both are prescribed by our algorithm, depending on two different specifications d and d'.

Consider the following basic object $(W,(J,b),E)$ of dimension 4 where

$$\begin{cases} W = \mathbb{A}^4 = \text{Spec } k[x_1,x_2,x_3,x_4], \\ J = \langle f \rangle \quad \text{with} \quad f = x_4^2 + x_3^3 + x_2x_3^2 + x_1^3, \\ b = 2, \\ E = \{H\} \quad \text{with} \quad H = \{x_3 = 0\}. \end{cases}$$

As in Remark 4-2 (ii), the basic object $(W,(J,b),E)$ defines a general basic object $(\mathcal{F},(W,E))$ with a 4-dimensional structure.

Also consider the following basic object $(X,(\mathfrak{a},c),F)$ of dimension 3 where

$$\begin{cases} X = \mathbb{A}^3 = \text{Spec } k[x_1,x_2,x_3], \\ \mathfrak{a} = \langle g \rangle \quad \text{with} \quad g = x_3^3 + x_2x_3^2 + x_1^3, \\ c = 2, \\ F = \{H_V\} \quad \text{with} \quad H_V = \{x_3 = 0\}. \end{cases}$$

From Giraud's Lemma (cf, Claim 3-4) and from a view point of looking at f as a polynomial in x_4, it follows immediately that, via the closed immersion of pairs $(X = \{x_4 = 0\}, F) \hookrightarrow (W,E)$, the basic object provides a (global) 3-dimensional chart to the general basic object $(\mathcal{F},(W,E))$.

Therefore, the general basic object has two different dimensions, namely 3 and 4, for its structure.

(i) Resolution of singularities of $(\mathcal{F},(W,E))$ with the 4-dimensional structure:

We apply our inductive algorithm to the basic object $(W,(J,b),E)$ of dimension 4.

Via direct computation, we see that

$$\begin{aligned} \text{Sing}(J,b) &= V(x_1,x_3,x_4), \\ \max\ t &= \max\ (w\text{-ord}, n) = (1,1), \\ \underline{\text{Max}}\ t &= V(x_1,x_2,x_3). \end{aligned}$$

Since $\text{codim}_W \underline{\text{Max}}\ t > 1$, accoprding to the algorithm, we proceed to construct a basic object $(W'',(J'',b''),E'')$ where

$$\begin{cases} W'' = W, \\ J'' = \overline{J} + \langle x_3^2 \rangle = J + \langle x_3^2 \rangle = \langle x_4^2 + x_1^3, x_3^2 \rangle, \\ b'' = b = 2, \\ E'' = \emptyset. \end{cases}$$

Now out of the basic object $(W'',(J'',b''),E'')$, knowing $x_4 \in \Delta(J'')$, we construct a basic object $(\widetilde{W''},(C(J''),b''!),\widetilde{E''})$ of dimension 3 where

$$\begin{cases} \widetilde{W''} = \{x_4 = 0\}, \\ C(J'') = \langle x_1^3, x_3^2 \rangle + \langle x_1^2, x_3 \rangle^2 = \langle x_1^3, x_3^2, x_1^2x_3 \rangle, \\ b''! = 2! = 2, \\ \widetilde{E''} = E''|_{\widetilde{W''}} = \emptyset. \end{cases}$$

Denote $(\widetilde{W''}, (C(J''), b''!), \widetilde{E''})$ by $(W^{(3)}, (J^{(3)}, b^{(3)}), E^{(3)})$ and the associated t-invariant by $t^{(3)}$ and the others by putting the superscript ${}^{(3)}$. Via direct computation, we see that

$$\begin{aligned}
\mathrm{Sing}(J^{(3)}, b^{(3)}) &= V(x_1, x_3), \\
\max\ t^{(3)} &= \max\ (w\text{-ord}^{(3)}, n^{(3)}) = (1, 0), \\
\underline{\mathrm{Max}}\ t^{(3)} &= V(x_1, x_3).
\end{aligned}$$

Since $\mathrm{codim}_{W^{(3)}} \underline{\mathrm{Max}}\ t^{(3)} > 1$, according to the algorithm, we proceed to construct a basic object $(W^{(3)''}, (J^{(3)''}, b^{(3)''}), E^{(3)''})$, which is nothing but $(W^{(3)}, (J^{(3)}, b^{(3)}), E^{(3)})$ itself in this case. Now out of the basic object $(W^{(3)''}, (J^{(3)''}, b^{(3)''}), E^{(3)''})$, knowing $x_3 \in \Delta(J^{(3)''})$, we construct a basic object $(\widetilde{W^{(3)''}}, (C(J^{(3)''}), b^{(3)''}!), \widetilde{E^{(3)''}})$ of dimension 2 where

$$\begin{cases}
\widetilde{W^{(3)''}} = \{x_3 = 0\}, \\
C(J^{(3)''}) = \langle x_1^3 \rangle + \langle x_1^2 \rangle^2 = \langle x_1^3 \rangle, \\
b^{(3)''}! = 2! = 2, \\
\widetilde{E^{(3)''}} = E^{(3)''}|_{\widetilde{W^{(3)''}}} = \emptyset.
\end{cases}$$

Denote $(\widetilde{W^{(3)''}}, (C(J^{(3)''}), b^{(3)''}!), \widetilde{E^{(3)''}})$ by $(W^{(2)}, (J^{(2)}, b^{(2)}), E^{(2)})$ and the associated t-invariant by $t^{(2)}$ and the others by putting the superscript ${}^{(2)}$. Via direct computation, we see that

$$\begin{aligned}
\mathrm{Sing}(J^{(2)}, b^{(3)}) &= V(x_1), \\
\max\ t^{(2)} &= \max\ (w\text{-ord}^{(2)}, n^{(2)}) = (\frac{3}{2}, 0), \\
\underline{\mathrm{Max}}\ t^{(2)} &= V(x_1).
\end{aligned}$$

Since this time

$$\mathrm{codim}_{W^{(2)}} \underline{\mathrm{Max}}\ t^{(2)} = 1,$$

according to the algorithm, we finally decide that the center Y_0 of the first transformation must be $Y_0 = R(1)(\underline{\mathrm{Max}}\ t^{(2)})$, i.e., via the inclusion $W^{(2)} = V(x_3, x_4) \subset W^{(4)} = W$ we have the description of the center

$$Y_0 = V(x_1, x_3, x_4) \subset W.$$

(ii) Resolution of singularities of $(\mathcal{F}, (W, E))$ with the 3-dimensional structure:

We apply our inductive algorithm to the basic object $(X, (\mathfrak{a}, c), F)$ of dimension 3.

Via direct computation, we see that

$$\begin{aligned}
\mathrm{Sing}(\mathfrak{a}, c) &= V(x_1, x_3), \\
\max\ t &= \max\ (w\text{-ord}, n) = (\frac{3}{2}, 1), \\
\underline{\mathrm{Max}}\ t &= V(x_1, x_2, x_3).
\end{aligned}$$

Since $\operatorname{codim}_X \underline{\mathrm{Max}}\ t > 1$, we have to proceed constructing the auxiliary basic objects. However, since we know that the center Y_0' that we take for the first transformation must satisfy the condition $Y_0' \subset \underline{\mathrm{Max}}\ t$, we conclude that via the inclusion $X = V(x_4) \subset W$ we have the description of the center

$$Y_0' = V(x_1, x_2, x_3, x_4) \subset W.$$

Comparing (i) and (ii), we see that the two sequences of resolution of singularities of the general basic object $(\mathcal{F}, (W, E))$, one with a 4-dimensional structure and the other with a 3-dimensional structure, have two different centers for the first transformations

$$Y_0 = V(x_1, x_3, x_4) \neq Y_0' = V(x_1, x_2, x_3, x_4) \subset W.$$

Therefore, the two sequences are obviously different.

Remark 11-4.

(i) Example 11-3 should not be confused with the fact that our algorithm for non-embedded resolution of singularities of a variety X does NOT depend on the choice of the number d which represents the common dimension of the ambient smooth varieties W_U, into which the open subsets U (in an open covering $\{U\}$ of X) are embedded (cf. Theorem 10-1).

It should be noted that the algorithm for non-embedded resolution of singularities of a variety X described in Encinas [1] is different from our algorithm in Chapter 10. Therefore, there is no contradiction between our Theorem 10-1 and the claim in Encinas [1] that the algorithm for non-embedded resolution of singularities of a variety X DOES depend on the choice of the number d which represents the common dimension of the ambient smooth varieties W_U, into which the open subsets U (in an open covering $\{U\}$ of X) are embedded.

(ii) Let $(\mathcal{F}, (W, E))$ be a general basic object with a d-dimensional structure, having charts $\{(\widetilde{W^\lambda}, (\mathfrak{a}, b^\lambda), \widetilde{E^\lambda})\}_{\lambda \in \Lambda}$ of $\dim \widetilde{W^\lambda} = d \quad \forall \lambda \in \Lambda$. Assume that $(\mathcal{F}, (W, E))$ has a d'-dimensional structure with $d' < d$.

Suppose that $(\mathcal{F}, (W, E))$ is simple in the sense that $(\widetilde{W^\lambda}, (\mathfrak{a}, b^\lambda), \widetilde{E^\lambda})$ is a simple basic object for all $\lambda \in \Lambda$.

Suppose further that $E = \emptyset$.

Then the sequence representing resolution of singularities of $(\mathcal{F}, (W, E))$ constructed according to our algorithm with the specified dimension of the structure being d, coincides with the sequence constructed according to our algorithm with the specified dimension of the structure being d'.

This can be seen as follows: Firstly observe that for a simple basic object (and their transformations) the invariants ord and w-ord coincide and, if further the boundary divisor is empty, then w-ord and the t-invariant coincide (until the maximum drops). Secondly, based upon the first observation, observe that the auxiliary general basic object of dimension d' that we construct in the process prescribed by our algorithm must coincide with the original general basic object with a d'-dimensional structure.

In Example 11-3, the general basic object $(\mathcal{F}, (W, E))$ is simple, arising from the simple basic object $(W, (J, b), E)$. Therefore, setting $E = \{H\} \neq \emptyset$ is essential

in order to get two different sequences of resolution of singularities, depending on $d = 4$ and $d' = 3$.

The author does not know an example of a general basic object $(\mathcal{F}, (W, E))$ with $E = \emptyset$, having structures of two different dimensions, for which our algorithm gives rise to two different sequences of resolution of singularities depending on the specified dimensions of the structures. Such a general basic object cannot be simple.

To reveal more ignorance, the author does not know an example of a basic object of dimension d which is not simple and which has a $(d-1)$-dimensional structure as a general basic object (or even if such a basic object exists).

Example 11-5 (The centers for non-embedded resolution may not be smooth.).

The following example of a sequence of embedded resolution of singularities of a variety $X \subset W$ (embedded as a closed subscheme in a smooth variety)

$$X = X_0 \subset W = W_0 \overset{\pi_1}{\leftarrow} X_1 \subset W_1 \overset{\pi_2}{\leftarrow} \cdots \overset{\pi_{l-1}}{\leftarrow} X_{l-1} \subset W_{l-1} \overset{\pi_l}{\leftarrow} X_l \subset W_l,$$

shows that the center $Y_{i-1} \subset W_{i-1}$, chosen by our inductive algorithm, is always smooth inside of the ambient variety W_{i-1} (cf. Theorem 7-1) by construction, but that

(α) the center Y_{i-1} may not be contained in the strict transform X_{i-1}, and/or

(β) the intersection of the center with the strict transform $Y_{i-1} \cap X_{i-1}$ may not be smooth.

(Remark that in such an example X must not be a hypersurface (cf. Remark 7-2 (ii)) and hence that the defining ideal $\mathcal{I}_X$ of X in W has to have two or more generators even locally.)

Since the sequence representing non-embedded resolution of singularities of X, constructed according to our inductive algorithm, is based upon the one representing embedded resolution of singularities (cf. Chapter 10), this example also shows that the centers for the sequence representing non-embedded resolution of singularities may not be smooth.

Let $X \subset W = \text{Spec } k[x, y, z, w]$ be a subvariety defined by the ideal

$$\mathcal{I}_X = \langle x^2 + y^2 + z^2 + w^2, x^6 + y^6 + z^6 + w^6 \rangle.$$

In order to obtain the sequence representing embedded resolution of singularities, we consider (cf. Chapter 10) a basic object $(W, (J, b), E)$ where

$$\begin{cases} W = \mathbb{A}^4 = \text{Spec } k[x, y, z, w] \\ J = \mathcal{I}_X \\ b = 1 \\ E = \emptyset. \end{cases}$$

It is straightforward to see that

$$w\text{-ord}(p) = \begin{cases} 2 & \text{if } p = 0 \\ 1 & \text{if } p \neq 0. \end{cases}$$

Since the center $Y_0 \subset W_0$ of the first transformation of basic objects

$$(W,(J,b),E) = (W_0,(J_0,b),E_0) \overset{\pi_1}{\leftarrow} (W_1,(J_1,b),E_1)$$

must be contained in the maximum locus of w-ord $=$ w-ord$_0$, i.e., $Y_0 \subset \underline{\mathrm{Max}}\ w\text{-ord}_0 = \{0\}$, we conclude that

$$Y_0 = \{0\}.$$

Consider the affine open subset of W_1 obtained by inverting x (We use the same letter W_1 for the affine open subset by abuse of notation.), with a system of regular parameters (x_1, y_1, z_1, w_1) with $x = x_1, y = xy_1, z = xz_1, w = xw_1$.

By direct computation we see that

$$\begin{aligned}
J_1 &= \langle x_1(1 + y_1^2 + z_1^2 + w_1^2), x_1^5(1 + y_1^6 + z_1^6 + w_1^6)\rangle \\
\overline{J_1} &= \langle 1 + y_1^2 + z_1^2 + w_1^2, x_1^4(1 + y_1^6 + z_1^6 + w_1^6)\rangle \\
\mathcal{I}_{X_1} &= \langle 1 + y_1^2 + z_1^2 + w_1^2, 1 + y_1^6 + z_1^6 + w_1^6\rangle
\end{aligned}$$

and hence that

$$\begin{aligned}
\mathrm{Sing}(J_1,b) &= V(x_1) \cup V(\langle 1 + y_1^2 + z_1^2 + w_1^2, 1 + y_1^6 + z_1^6 + w_1^6\rangle) \\
w\text{-ord}_1(p_1) &= \begin{cases} 0 & \text{if } p_1 \in \mathrm{Sing}(J_1,b) \setminus V(1 + y_1^2 + z_1^2 + w_1^2) \\ 1 & \text{if } p_1 \in \mathrm{Sing}(J_1,b) \cap V(1 + y_1^2 + z_1^2 + w_1^2) \end{cases} \\
t_1(p_1) &= \begin{cases} (0,1) \text{ if } p_1 \in \mathrm{Sing}(J_1,b) \setminus V(1 + y_1^2 + z_1^2 + w_1^2) \\ (1,0) \text{ if } p_1 \in (\mathrm{Sing}(J_1,b) \cap V(1 + y_1^2 + z_1^2 + w_1^2)) \setminus V(x_1) \\ (1,1) \text{ if } p_1 \in \mathrm{Sing}(J_1,b) \cap V(1 + y_1^2 + z_1^2 + w_1^2) \cap V(x_1). \end{cases}
\end{aligned}$$

Therefore, we have

$$\begin{aligned}
&\max\ t_1 = (1,1) \\
&\underline{\mathrm{Max}}\ t_1 = V(x_1) \cap V(1 + y_1^2 + z_1^2 + w_1^2).
\end{aligned}$$

Now since

$$\mathrm{codim}_{W_1} \underline{\mathrm{Max}}\ t_1 > 1,$$

our inductive algorithm tells us (cf. Theorem 5-1) to construct a basic object $(W_1'', (J_1'', b''), E'')$ (cf. Lemma 5-3 and lemma 5-4) where

$$\begin{cases}
W_1'' = W_1, \\
J_1'' = \overline{J_1} + \langle x_1\rangle = \langle 1 + y_1^2 + z_1^2 + w_1^2, x_1^4(1 + y_1^6 + z_1^6 + w_1^6), x_1\rangle, \\
b'' = b = 1, \\
E_1'' = E_1^+ = \emptyset.
\end{cases}$$

The basic object $(W_1'', (J_1'', b''), E'')$ has a $3(= 4-1)$-dimensional structure, whose chart is given by

$$((W_1'')_h, (C(J_1''), b''!), (E_1'')_h) = (\{x_1 = 0\}, (\langle 1 + y_1^2 + z_1^2 + w_1^2\rangle, 1), \emptyset).$$

We see that

$$\mathrm{Sing}(C(J_1''), b''!) = V(1 + y_1^2 + z_1^2 + w_1^2)$$

and that

$$w\text{-ord}(q) = 1, t(q) = (1, 0) \quad \forall q \in \mathrm{Sing}(C(J_1''), b''!).$$

Therefore, we have

$$R(1)(\underline{\mathrm{Max}}\ t_1''^{(3)}) = \underline{\mathrm{Max}}\ t_1''^{(3)} = \mathrm{Sing}(C(J_1''), b''!) = V(1 + y_1^2 + z_1^2 + w_1^2).$$

Therefore, according to our inductive algorithm, the center $Y_1 \subset W_1$ for the second transformation of basic objects

$$(W_1, (J_1, b), E_1) \stackrel{\pi_2}{\leftarrow} (W_2, (J_2, b), E_2)$$

has to be taken so that

$$Y_1 = V(x_1, 1 + y_1^2 + z_1^2 + w_1^2) \subset W_1.$$

Although Y_1 itself is smooth, the intersection with the strict transform is described by

$$Y_1 \cap X_1 = V(x_1, 1 + y_1^2 + z_1^2 + w_1^2, 1 + y_1^6 + z_1^6 + w_1^6),$$

whose singular locus is characterized by the condition

$$\mathrm{rank} \begin{bmatrix} 2y_1 & 2z_1 & 2w_1 \\ 6y_1^5 & 6z_1^5 & 6w_1^5 \end{bmatrix} < 2$$

and hence contains a point, e.g.,

$$(x_1, y_1, z_1, w_1) = (0, 0, 0, i).$$

Therefore, $Y_1 \cap X_1$ is SINGULAR.

Remark 11-6.

In general, we observe (cf. the footnote to Main Theme 0-1, Remark 10-2 (ii)) that

(α) the center Y_{i-1} may not be contained in the strict transform X_{i-1}, and/or

(β) the intersection of the center with the strict transform $Y_{i-1} \cap X_{i-1}$ may not be smooth or even reduced.

We leave it as an exercise to the reader to produce an example where $Y_{i-1} \cap X_{i-1}$ is actually non-reduced.

Example 11-7 (Resolution of singularities of a monomial basic object.).

Finally, we give an example demonstrating how to construct a sequence representing resolution of singularities of a monomial basic object, explaining how the tie breaker works and how the convention in Definition 1-8 (iii) works. It could be of help to the reader trying to fill in the proof to Proposition 2-5.

We start with a monomial basic object $B = (W, (J, b), E)$ where

$$\begin{cases} W = \mathbb{A}^2 = \text{Spec } k[x, y] \\ J = I(H_1)^3 I(H_2)^3 \\ b = 2 \\ E = \{H_1, H_2\}, \end{cases}$$

where $H_1 = V(x)$ and $H_2 = V(y)$.

It is straightforward to see that

$$\begin{aligned} \text{Sing}(J, b) &= H_1 \cup H_2, \\ \Gamma_{B_0}(p) &= \begin{cases} (-1, \frac{3}{2}, (1, 0)) \text{ if } p \in H_1 \setminus H_2 \\ (-1, \frac{3}{2}, (2, 0) \text{ if } p \in H_2. \end{cases} \end{aligned}$$

Therefore, we have

$$\begin{aligned} &\max \ \Gamma_{B_0} = (-1, \frac{3}{2}, (2, 0)) \\ &\underline{\text{Max}} \ \Gamma_{B_0} = H_2. \end{aligned}$$

Observe that the indices of H_1 and H_2 work as a tiebreaker.

According to our algorithm, we choose $Y_0 = \underline{\text{Max}}\ \Gamma_{B_0} = H_2$ to be the center for the first transformation

$$(W, (J, b), E) = B = B_0 = (W_0, (J_0, b), E_0) \quad \stackrel{\pi_1}{\leftarrow} \quad B_1 = (W_1, (J_1, b), E_1).$$

Since Y_0 is a divisor, $W_0 \stackrel{\pi_1}{\leftarrow} W_1$ is an isomorphism, whereas, according to the convention in Definition 1-8 (iii), H_2 is now called H_3 and hence we have

$$J_1 = I(H_1)^3 I(H_3) \quad \text{and} \quad E_1 = \{H_1, H_3\}.$$

We see that

$$\begin{aligned} \text{Sing}(J_1, b) &= H_1, \\ \Gamma_{B_1}(p) &= (-1, \frac{3}{2}, (1, 0)) \quad \forall p \in H_1. \end{aligned}$$

Therefore, we have

$$\begin{aligned} &\max \ \Gamma_{B_1} = (-1, \frac{3}{2}, (1, 0)) \\ &\underline{\text{Max}} \ \Gamma_{B_1} = H_1. \end{aligned}$$

According to our algorithm, we choose $Y_1 = \underline{\text{Max}}\ \Gamma_{B_1} = H_1$ to be the center for the second transformation

$$B_1 = (W_1, (J_1, b), E_1) \quad \stackrel{\pi_2}{\leftarrow} \quad B_2 = (W_2, (J_2, b), E_2).$$

Since Y_1 is a divisor, $W_1 \overset{\pi_2}{\leftarrow} W_2$ is an isomorphism, whereas, according to the convention in Definition 1-8 (iii) again, H_1 is now called H_4 and hence we have

$$J_2 = I(H_3)I(H_4) \quad \text{and} \quad E_2 = \{H_3, H_4\}.$$

We see that

$$\begin{aligned} \operatorname{Sing}(J_2, b) &= H_3 \cap H_4, \\ \Gamma_{B_2}(p) &= (-2, 1, (4,3)) \quad \forall p \in H_3 \cap H_4. \\ &\text{(Note that } H_1 \cap H_4 \text{ consists of a point.)} \end{aligned}$$

Therefore, we have

$$\begin{aligned} &\max\ \Gamma_{B_2} = (-2, 1, (4,3)) \\ &\underline{\mathrm{Max}}\ \Gamma_{B_2} = H_3 \cap H_4. \end{aligned}$$

According to our algorithm, we choose $Y_2 = \underline{\mathrm{Max}}\ \Gamma_{B_2} = H_3 \cap H_4$ to be the center for the third transformation

$$B_2 = (W_2, (J_2, b), E_2) \quad \overset{\pi_3}{\leftarrow} \quad B_3 = (W_3, (J_3, b), E_3).$$

We observe then that

$$\operatorname{Sing}(J_3, b) = \emptyset$$

and the sequence stops here achieving resolution of singularities.

Note that in the process we have the invariant Γ strictly decreasing

$$\max\ \Gamma_{B_0} = (-1, \frac{3}{2}, (2,0)) > \max\ \Gamma_{B_1} = (-1, \frac{3}{2}, (1,0)) > \max\ \Gamma_{B_2} = (-2, 1, (4,3)).$$

REFERENCES

We should emphasize that the purpose of these notes is to give a self-contained exposition of the results of our seminar on the specified inductive algorithm by Encinas and Villamayor, and not to write a treatise on the subject of resolution of singularities in general. Accordingly, our references are very restricted and limited to those which are directly related to the inductive algorithm we concentrate our focus on and which happen to fall upon the eyes of the author, and we do not even pretend to try to list a part of the vast literature which may probably be connected to the algorithm in an explicit or implicit way. (For example, we list no references to the recent development of "weak resolution" by Abramovich-de Jong, Bogomolov-Pantev and others, based upon the work of de Jong on alterations.) The readers are encouraged to look at the references of the papers listed below for those references that we miss here and beyond.

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